Same instructions as Mission 1. Thanks!

**Problem 41** Suppose $T : V \times V \times V^* \to \mathbb{R}$ is a multilinear map. If $\beta = \{e_1, \dots, e_n\}$ is basis for $V$ and $\theta^1, \dots, \theta^n$ is the dual basis for $V^*$ then find $C_{ij}{}^k \in \mathbb{R}$ for which $T = \sum_{i,j,k} C_{ij}{}^k \theta^i \otimes \theta^j \otimes v_k$.

**Problem 42** Renteln Exercise 2.2 page 36. ( contraction of symmetric and antisymmetric gives zero)

**Problem 43** Renteln Exercise 2.5 page 41. ($\alpha \wedge \beta = (-1)^{|\alpha|}\beta \wedge \alpha$ see my notes for help )

**Problem 44** Renteln Exercise 2.15 page 50. (Cartan's Lemma)

**Problem 45** Renteln Exercise 2.17 page 50. (Hodge duality on $\mathbb{R}^n$ with Euclidean metric)

**Problem 46** Let $\omega_{\langle a,b,c\rangle} = ae^1 + be^2 + ce^3$ and $\Phi_{\langle a,b,c\rangle} = ae^2 \wedge e^3 + be^3 \wedge e^1 + ce^1 \wedge e^2$ notice the Hodge dual with respect to the Euclidean metric on $\mathbb{R}^3$ gives $\star\omega_{\langle a,b,c\rangle} = \Phi_{\langle a,b,c\rangle}$ and $\star 1 = e^1 \wedge e^2 \wedge e^3$ and $\star^2 = 1$. Calculate the following:

- **(a.)** $\omega_{\vec{A}} \wedge \omega_{\vec{B}}$ and write your answer in terms of $\Phi$ of a well-known vector product from Calculus III,
- **(b.)** $\star(\omega_{\vec{A}} \wedge \omega_{\vec{B}} \wedge \omega_{\vec{C}})$ and explain the geometric significance of this quantity which hopefully you saw in Calculus III.

**Problem 47** Consider $\mathbb{R}^4$ and define $\omega_{\langle a,b,c,d\rangle} = ae^1 + be^2 + ce^3 + de^4$. Let $\star$ denote the Hodge dual with respect to the Euclidean metric and define:

$$\vec{A} \times \vec{B} \times \vec{C} = \omega^{-1}\left(\star\left(\omega_{\vec{A}} \wedge \omega_{\vec{B}} \wedge \omega_{\vec{C}}\right)\right)$$

for $\vec{A}, \vec{B}, \vec{C} \in \mathbb{R}^4$.

- **(a.)** Prove $\vec{A} \times \vec{B} \times \vec{C} \in \{\vec{A}, \vec{B}, \vec{C}\}^\perp$
- **(b.)** Given $\vec{A}, \vec{B}, \vec{C}$ are orthogonal, show $\|\vec{A} \times \vec{B} \times \vec{C}\| = \|\vec{A}\|\|\vec{B}\|\|\vec{C}\|$.
- **(c.)** Can you write a formula for this generalized cross-product using a nonsense determinant in remembrance of calculus III ?

**Problem 48** Let $\mathcal{I}(p, n)$ denote the set of all strictly increasing $p$-tuples of indices taken from $\{1, 2, \dots, n\}$. Show that if $\beta = \{e_1, \dots, e_n\}$ is a basis for $V$ then $\Lambda^p V$ has basis $\beta^p = \{e_I \mid I \in \mathcal{I}(p, n)\}$ where the **multi-index** notation $I = (i_1, \dots, i_p)$ implicits $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$.