Same instructions as Mission 1. Thanks!

Problem 49 Let $F(x,y,z) = (\cos(x), \sin(xy))$. Calculate the push-forward of $X = a\partial_x + b\partial_y + c\partial_z$.

Problem 50 Observe $\chi = (\theta, \phi)$ gives a coordinate chart on $S_2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$. The inverse of this chart is given by the patch $\chi^{-1}(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Calculate the push-forward under χ^{-1} of ∂_θ and ∂_ϕ in terms of the standard coordinate derivations $\partial_x, \partial_y, \partial_z$ the Cartesian coordinate system (x,y,z) for $\mathbb{R}^3$.

Problem 51 Renteln Exercise 3.19 page 74-75. (spherical frame from derivation viewpoint)

Problem 52 Renteln Exercise 3.20 page 77-78. (Lie bracket of vector fields)

Problem 53 Renteln Exercise 3.22 page 84. (derivative of vector field not tensorial)

Problem 54 Renteln Exercises 3.25 and 3.26 page 90. (exterior differentiation)

Problem 55 In $\mathbb{R}^4$ with metric $\eta = Diag(-1,1,1,1)$ I describe in my notes how Hodge duality introduces certain signs. The basic idea is very much like the simpler context of $\mathbb{R}^3$. Because I use the metric which agrees with the euclidean metric on x, y, z components the work and flux-form correspondences naturally generalize: a general one-form on $\mathbb{R}^4_{txyz}$ space has the form:

$$\alpha = \alpha_0 dt + \alpha_1 dx + \alpha_2 dy + \alpha_3 dz = \alpha_0 dt + \omega_{\vec{\alpha}}.$$

Notice, I am encouraging the notation $\vec{\alpha}$ for the spatial vector piece of the one-form α. No such simple correspondence is possible for a generic two-form since it has six independent components:

$$\begin{aligned}\beta &= F_1 dt\wedge dx + F_2 dt\wedge dy + F_3 dt\wedge dz + G_1 dy\wedge dz + G_2 dz\wedge dx + G_3 dx\wedge dy \\ &= dt\wedge\omega_{\vec{F}} + \Phi_{\vec{G}}\end{aligned}$$

Clearly the formula in terms of the work and flux-form correspondance will make it easier for us to follow calculus and algebra for β. Next, a three-form has the general form:

$$\begin{aligned}\gamma &= G_0 dx\wedge dy\wedge dz + G_1 dt\wedge dy\wedge dz + G_2 dt\wedge dz\wedge dx + G_3 dt\wedge dx\wedge dy \\ &= G_0 dx\wedge dy\wedge dz + dt\wedge\Phi_{\vec{\gamma}}\end{aligned}$$

where I am encouraging use of the notation $\vec{\gamma} = \langle G_1, G_2, G_3\rangle$ to emphasize the correspondence between spatial 3-vectors and those components of γ. Continuting, there is just one 4-form:

$$\zeta = f dt\wedge dx\wedge dy\wedge dz.$$

Please notice that all the coefficients of the forms are in fact 0-forms on $\mathbb{R}^4$, that is, functions of t, x, y, z. This introduces time derivative terms in the formulas you are to find below. Use the notation given above to calculate:

(a.) df where f is a real-valued function on $\mathbb{R}^4_{txyz}$.

(b.) $d\alpha$

(c.) $d\beta$

(d.) $d\gamma$

(e.) $d\zeta$

Problem 56 Again, using the notation introduced in the previous problem, find the explicit (and as nice as possible) formulas for:

(a.) $\star f$ where f is a real-valued function on $\mathbb{R}^4_{txyz}$.

(b.) $\star\alpha$

(c.) $\star\beta$

(d.) $\star\gamma$

(e.) $\star\zeta$