Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 Your signature below indicates you have:

(a.) I have read much of Cook's Chapter 1, 2 and 3:________________.

Problem 2 Let $T(x,y) = (3x+y, x+y, y)$ define function from $\mathbb{R}^2$ to $\mathbb{R}^3$. Show this function is linear by writing its formula as $T(v) = Av$ for appropriate matrix A. In other words, find $[T]$. Determine if T is surjective. Determine if T is injective.

Problem 3 Find a real basis β for antisymmetric 3×3 real matrices. Also, give the formula for Φ_β.

Problem 4 Find a real basis β for symmetric 2×2 complex matrices. Also, give the formula for Φ_β.

Problem 5 Let $T : P_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ be defined as follows:

$$T(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f''(0) & -f(0) \end{bmatrix}$$

If $\beta = \{1, x, x^2\}$ serves as a basis for $P_2(\mathbb{R})$ and $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ serves as the basis for $\mathbb{R}^{2\times 2}$ then find $[T]_{\beta,\gamma}$. Also, calculate $\ker(T)$. Is T injective? Is T surjective?

Problem 6 For the map of the previous problem we can choose a different codomain to define a new function $\overline{T} : P_2(\mathbb{R}) \to W$ where W is the set of traceless ($\text{trace}(A) = 0$ for $A \in W$) 2×2 matrices. Once again, define

$$\overline{T}(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f''(0) & -f(0) \end{bmatrix}$$

If $\beta = \{1, x, x^2\}$ and $\overline{\gamma} = \{E_{11} - E_{22}, E_{12}, E_{21}\}$ then find $[\overline{T}]_{\beta,\overline{\gamma}}$. Is T surjective ?

Problem 7 I defined $A \oplus B$ and $A \otimes B$ in the lecture notes. Is it true that $A \otimes (B+C) = (A \otimes B) + (A \otimes C)$? Is it true that $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$?

Problem 8 Calculate $A \otimes B$ and $A \oplus B$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Calculate the determinant and trace of $A, B, A \oplus B$ and $A \otimes B$. Is there a pattern amongst these quantities?

Problem 9 Let $\eta(A,B) = \text{trace}(A^T B)$ for all $A, B \in \mathbb{R}^{n\times n}$. Show that η is an inner product on $\mathbb{R}^{n\times n}$. This shows that $||A|| = \sqrt{\text{trace}(A^T A)}$ is a norm as it is simply the norm induced from η. Also, calculate $||A||$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

you should see this verifies my claim that $||A|| = ||\Phi_\beta(A)||$ *, that is,* $||A||$ *is just the length of* A *when rewritten as a single vector stringing out row after row into a big vector. I should mention again,* $||A||$ *as we define it here is the* **Frobenius norm**

Problem 10 Let $A, B \in \mathbb{R}^{n\times n}$ and $||\cdot||$ denote the Frobenius norm. Show that $||AB|| \leq ||A||\,||B||$. (perhaps this page of notes is helpful)

Problem 11 Show that $||(x,y)||_\infty = \max\{|x|, |y|\}$ defines a norm on $\mathbb{R}^2$.

Problem 12 Suppose V, W are real normed linear spaces. We say f is continuous at $a \in V$ if $\lim_{x\to a} f(x) = f(a)$. Furthermore, f is continuous on $U \subseteq V$ if f is continuous at each point in U. Let $f : V \to W$ then

$$f^{-1}(V) = \{x \in S \mid f(x) \in V\}$$

is the **inverse image** of V under f. Show: $f : U \to W$ is continuous on U if and only if the inverse image of each open set in W under f is an open set in U. Note, by default, we consider the emptyset $\emptyset$ an open set.

topology is the study of continuity in the abstract. This equivalence shows us that we can define continuity of a function without direct reliance on some concept of distance. It suffices to define which sets in the domain and codomain are open. A topological space is simply a set paired with a family of all the open sets which means any set can be given a topology. There are many topological spaces which are not normed linear spaces. I'll leave further exposition of that for your topology course.

Problem 13 Let V be a real inner product space with inner product $\langle , \rangle : V \times V \to \mathbb{R}$. Veryify that the **induced norm** $||x|| = \sqrt{\langle x, x\rangle}$ is indeed a norm.

Problem 14 Let V be a real vector space for which there exist two norms $||\cdot||_1$ and $||\cdot||_2$. Further, assume $||\cdot||_1$ and $||\cdot||_2$ are equilvalent norms. That is, assume there exist nonzero constants m, M for which

$$m||x||_1 \leq ||x||_2 \leq M||x||_1.$$

Show that $U \subset V$ is open with respect to $||\cdot||_1$ if and only if $U \subset V$ is open with respect to $||\cdot||_2$.

I did not ask you to prove the norms are equivalent. If you would like to see more about the equivalence of norms then you can read this wikipedia article or this math stackexchange Q and A

Problem 15 Show $\mathbb{R}^{m\times n}$ is a complete space. Assume it is already known that $\mathbb{R}^{m\times n}$ is a normed linear space with respect to the Frobenius norm $||A|| = \text{trace}(A^TA)$.

Problem 16 Show $\{A\}$ is a closed set in $\mathbb{R}^{m\times n}$.

Problem 17 Suppose P is a paralellogram in $\mathbb{R}^3$ in the octant with positive coordinates. Furthermore, define $P = \{\vec{r}_o + u\vec{A} + v\vec{B} \mid (u, v) \in [0, 1]^2\}$.

(a.) find $Area(P)$.

(b.) define $L_{ij}(\vec{v}) = (\vec{v} \bullet \widehat{x}_i)\,\widehat{x}_i + (\vec{v} \bullet \widehat{x}_j)\,\widehat{x}_j$ and prove using properties of dot-products that L_{ij} is a linear transformation.

(c.) assume that linear transformations map parallelograms to lines, parallelograms or points and use this presupposition to establish the following equation:

$$Area(P)^2 = Area(L_{12}(P))^2 + Area(L_{31}(P))^2 + Area(L_{23}(P))^2$$

Problem 18 Let $A_3 = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$. Find a basis $\{f_1, f_2, f_3\}$ for A_3 by writing $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and studying the condition $A^T = -A$. You should find $\dim(A_3) = 3$.

Bonus: *study the isomorphism $\Phi : A_3 \rightarrow \mathbb{R}^3$ defined by linearly extending $\Phi(f_i) = e_i$, if we think of antisymmetric 3×3 matrices as vectors then what is the geometric meaning of matrix multiplication for A_3?*