

Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 Your signature below indicates you have:

(a.) I have read much of Cook's Chapter 1, 2 and 3: _____.

Problem 2 Let $T(x, y) = (3x + y, x + y, y)$ define function from \mathbb{R}^2 to \mathbb{R}^3 . Show this function is linear by writing its formula as $T(v) = Av$ for appropriate matrix A . In other words, find $[T]$. Determine if T is surjective. Determine if T is injective.

Problem 3 Find a real basis β for antisymmetric 3×3 real matrices. Also, give the formula for Φ_β .

Problem 4 Find a real basis β for symmetric 2×2 complex matrices. Also, give the formula for Φ_β .

Problem 5 Let $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^{2 \times 2}$ be defined as follows:

$$T(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f''(0) & -f(0) \end{bmatrix}$$

If $\beta = \{1, x, x^2\}$ serves as a basis for $P_2(\mathbb{R})$ and $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ serves as the basis for $\mathbb{R}^{2 \times 2}$ then find $[T]_{\beta, \gamma}$. Also, calculate $\ker(T)$. Is T injective? Is T surjective?

Problem 6 For the map of the previous problem we can choose a different codomain to define a new function $\bar{T} : P_2(\mathbb{R}) \rightarrow W$ where W is the set of traceless (trace(A) = 0 for $A \in W$) 2×2 matrices. Once again, define

$$\bar{T}(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f''(0) & -f(0) \end{bmatrix}$$

If $\beta = \{1, x, x^2\}$ and $\bar{\gamma} = \{E_{11} - E_{22}, E_{12}, E_{21}\}$ then find $[\bar{T}]_{\beta, \bar{\gamma}}$. Is T surjective?

Problem 7 I defined $A \oplus B$ and $A \otimes B$ in the lecture notes. Is it true that $A \otimes (B+C) = (A \otimes B) + (A \otimes C)$? Is it true that $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$?

Problem 8 Calculate $A \otimes B$ and $A \oplus B$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Calculate the determinant and trace of $A, B, A \oplus B$ and $A \otimes B$. Is there a pattern amongst these quantities?

Problem 9 Let $\eta(A, B) = \text{trace}(A^T B)$ for all $A, B \in \mathbb{R}^{n \times n}$. Show that η is an inner product on $\mathbb{R}^{n \times n}$. This shows that $\|A\| = \sqrt{\text{trace}(A^T A)}$ is a norm as it is simply the norm induced from η . Also, calculate $\|A\|$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

you should see this verifies my claim that $\|A\| = \|\Phi_\beta(A)\|$, that is, $\|A\|$ is just the length of A when rewritten as a single vector stringing out row after row into a big vector. I should mention again, $\|A\|$ as we define it here is the Frobenius norm

Problem 10 Let $A, B \in \mathbb{R}^{n \times n}$ and $\|\cdot\|$ denote the Frobenius norm. Show that $\|AB\| \leq \|A\| \|B\|$. (perhaps this page of notes is helpful)

Problem 11 Show that $\|(x, y)\|_\infty = \max\{|x|, |y|\}$ defines a norm on \mathbb{R}^2 .

Problem 12 Suppose V, W are real normed linear spaces. We say f is continuous at $a \in V$ if $\lim_{x \rightarrow a} f(x) = f(a)$. Furthermore, f is continuous on $U \subseteq V$ if f is continuous at each point in U . Let $f : V \rightarrow W$ then

$$f^{-1}(V) = \{x \in S \mid f(x) \in V\}$$

is the **inverse image** of V under f . Show: $f : U \rightarrow W$ is continuous on U if and only if the inverse image of each open set in W under f is an open set in U . Note, by default, we consider the emptyset \emptyset an open set.

topology is the study of continuity in the abstract. This equivalence shows us that we can define continuity of a function without direct reliance on some concept of distance. It suffices to define which sets in the domain and codomain are open. A topological space is simply a set paired with a family of all the open sets which means any set can be given a topology. There are many topological spaces which are not normed linear spaces. I'll leave further exposition of that for your topology course.

Problem 13 Let V be a real inner product space with inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$. Verify that the **induced norm** $\|x\| = \sqrt{\langle x, x \rangle}$ is indeed a norm.

Problem 14 Let V be a real vector space for which there exist two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Further, assume $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms. That is, assume there exist nonzero constants m, M for which

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

Show that $U \subset V$ is open with respect to $\|\cdot\|_1$ if and only if $U \subset V$ is open with respect to $\|\cdot\|_2$.

I did not ask you to prove the norms are equivalent. If you would like to see more about the equivalence of norms then you can read this wikipedia article or this math stackexchange Q and A

Problem 15 Show $\mathbb{R}^{m \times n}$ is a complete space. Assume it is already known that $\mathbb{R}^{m \times n}$ is a normed linear space with respect to the Frobenius norm $\|A\| = \text{trace}(A^T A)$.

Problem 16 Show $\{A\}$ is a closed set in $\mathbb{R}^{m \times n}$.

Problem 17 Suppose P is a parallelogram in \mathbb{R}^3 in the octant with positive coordinates. Furthermore, define $P = \{\vec{r}_o + u\vec{A} + v\vec{B} \mid (u, v) \in [0, 1]^2\}$.

(a.) find $\text{Area}(P)$.

(b.) define $L_{ij}(\vec{v}) = (\vec{v} \cdot \hat{x}_i) \hat{x}_i + (\vec{v} \cdot \hat{x}_j) \hat{x}_j$ and prove using properties of dot-products that L_{ij} is a linear transformation.

(c.) assume that linear transformations map parallelograms to lines, parallelograms or points and use this presupposition to establish the following equation:

$$\text{Area}(P)^2 = \text{Area}(L_{12}(P))^2 + \text{Area}(L_{31}(P))^2 + \text{Area}(L_{23}(P))^2$$

Problem 18 Let $A_3 = \{A \in \mathbb{R}^{3 \times 3} \mid A^T = -A\}$. Find a basis $\{f_1, f_2, f_3\}$ for A_3 by writing $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and studying the condition $A^T = -A$. You should find $\dim(A_3) = 3$.

Bonus: study the isomorphism $\Phi : A_3 \rightarrow \mathbb{R}^3$ defined by linearly extending $\Phi(f_i) = e_i$, if we think of antisymmetric 3×3 matrices as vectors then what is the geometric meaning of matrix multiplication for A_3 ?

[P2] $T(x, y) = \begin{bmatrix} 3x + y \\ x + y \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \therefore [T] = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

- the columns of $[T]$ are LI $\Rightarrow \ker(T) = \{0\} \Rightarrow T$ injective.
- $\dim(\text{Range}(T)) = 2 < 3 = \dim(\mathbb{R}^3) \Rightarrow T$ not surjective.

[P3] Let $A_3 = \{A \in \mathbb{R}^{3 \times 3} \mid A^T = -A\}$. Let $A \in A_3$,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = - \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \begin{array}{l} \rightarrow a = -a \Rightarrow a = 0 \\ \rightarrow b = -d \\ \rightarrow c = -g \\ \rightarrow e = -e \Rightarrow e = 0 \\ \rightarrow f = -h \\ \rightarrow i = -i \Rightarrow i = 0 \end{array}$$

$$A = \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix} = b \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_{12} - E_{21}} + c \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{E_{13} - E_{31}} + f \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{E_{23} - E_{32}}$$

Then $\beta = \{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is basis
for A_3 and for $A \in A_3$,

$$\Phi_\beta \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_A = (b, c, f). \quad \text{formula for } \Phi_\beta$$

[P4] $S_2(\mathbb{C}) = \{A \in \mathbb{C}^{2 \times 2} \mid A^T = A\}$. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow b = c$.

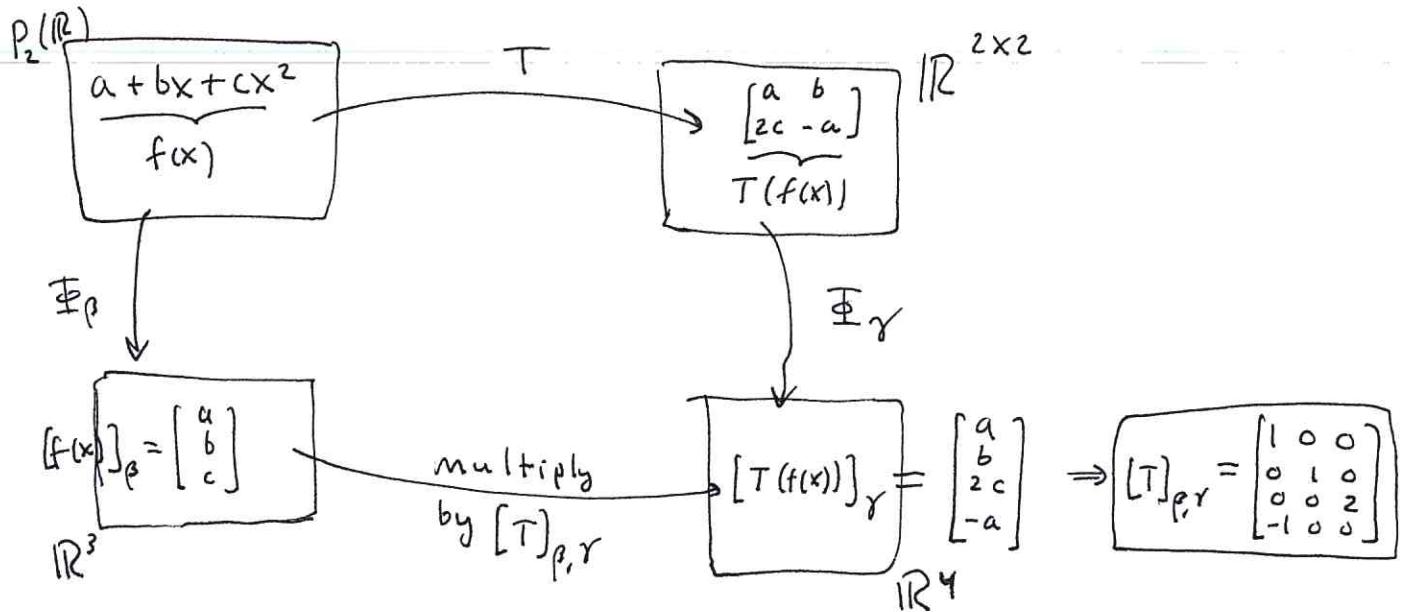
thus $Z = X + iY \in S_2(\mathbb{C})$ has form,

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_2 & Z_3 \end{bmatrix} = \left[\begin{array}{c|c} X_1 + iY_1 & X_2 + iY_2 \\ \hline X_2 + iY_2 & X_3 + iY_3 \end{array} \right] \hookrightarrow \boxed{\Phi_\beta(Z) = (X_1, Y_1, X_2, Y_2, X_3, Y_3)} \\ \notin \beta = \{E_{11}, iE_{11}, E_{12} + E_{21}, i(E_{12} + E_{21}), E_{22}, iE_{22}\}}$$

$$\boxed{P5} \quad T: P_2(\mathbb{R}) \rightarrow \mathbb{R}^{2 \times 2}$$

$$T(f(x)) = \begin{bmatrix} f(0) & f'(0) \\ f''(0) & -f(0) \end{bmatrix}$$

$$T(a + bx + cx^2) = \begin{bmatrix} a & b \\ 2c & -a \end{bmatrix}$$



Of course, we can also calculate w/o diagram,

$$\begin{aligned} [T]_{\beta, \gamma} &= \left[[T(1)]_\gamma \mid [T(x)]_\gamma \mid [T(x^2)]_\gamma \right] \\ &= \left[\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right]_\gamma \mid \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right]_\gamma \mid \left[\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix} \right]_\gamma \right] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

$$\text{Ker}(T) = \{ f(x) \in P_2 \mid T(f(x)) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \} \Rightarrow f(x) \in \text{Ker}(T)$$

has $f(0) = f'(0) = f''(0) = 0$. But $f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$
hence $f(x) = 0 \Rightarrow \boxed{\text{Ker}(T) = \{0\}}$.

- $\text{Ker}(T) = 0 \Rightarrow T$ is injective.

- $\dim(\text{Range}(T)) = 3$ as the columns of $[T]_{\beta, \gamma}$ are clearly LI
but $\dim(\mathbb{R}^{2 \times 2}) = 4 > 3 \Rightarrow T$ not surjective.

Remark: Let me review why T injective $\Leftrightarrow \text{Ker}(T) = \{0\}$.

Let $T: V \rightarrow W$ be linear transformation. Suppose

T is injective. Notice $T(o+o) = T(o) + T(o)$

but $o+o = o$ thus $T(o) = T(o) + T(o) \Rightarrow T(o) = o$

If $x \in \text{Ker}(T)$ then $T(x) = o$ hence by comparison to

$T(o) = o$ we find by injectivity that $x = o \Rightarrow \text{Ker } T = \{o\}$.

(Conversely, if $\text{Ker}(T) = \{o\}$ then consider $T(x) = T(y)$)

$$\Rightarrow T(x-y) = o \Rightarrow x-y \in \text{Ker}(T) = \{o\} \therefore x = y = o$$

and we conclude $x = y$ hence T is injective.

————— //

- the little argument above allows us to study injectivity merely by evaluating the structure of the $\text{Ker}(T)$.

[P6] $\bar{T}: P_2(\mathbb{R}) \rightarrow \bar{W} = \{ A \in \mathbb{R}^{2 \times 2} \mid \text{trace}(A) = 0 \}, \beta = \{1, x, x^2\}$,

Let $\bar{\gamma} = \{E_{11} - E_{22}, E_{12}, E_{21}\}$ serve as basis for \bar{W} . Find $[\bar{T}]_{\beta, \bar{\gamma}}$

$$[\bar{T}]_{\beta, \bar{\gamma}} = \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{\bar{\gamma}} \middle| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\bar{\gamma}} \middle| \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}_{\bar{\gamma}} \right] : \Phi_{\bar{\gamma}} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = (a, b, c)$$

$$[\bar{T}]_{\beta, \bar{\gamma}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Now, $\bar{T}: P_2(\mathbb{R}) \rightarrow \bar{W}$ is a linear transformation between V -spaces of EQUAL DIMENSION. It follows

\bar{T} is injective $\Leftrightarrow T$ is surjective. Observe

$$\text{rank}([\bar{T}]_{\beta, \bar{\gamma}}) = 3 \Rightarrow \bar{T} \text{ is surjective}$$

Moreover, we also know \bar{T} is injective in the special case.

Moral of Story: can swap codomain for range to make surjective

P7

Let $B, C \in \mathbb{R}^{P \times q}$ and $A \in \mathbb{R}^{m \times n}$

$$\begin{aligned}
 A \otimes (B+C) &= \begin{bmatrix} A_{11}(B+C) & \cdots & A_{1n}(B+C) \\ \vdots & \ddots & \vdots \\ A_{m1}(B+C) & \cdots & A_{mn}(B+C) \end{bmatrix} \\
 &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix} + \begin{bmatrix} A_{11}C & \cdots & A_{1n}C \\ \vdots & \ddots & \vdots \\ A_{m1}C & \cdots & A_{mn}C \end{bmatrix} \\
 &= (A \otimes B) + (A \otimes C). \quad (\text{true})
 \end{aligned}$$

Likewise, consider,

$$\begin{aligned}
 A \otimes (B \oplus C) &= A \otimes \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \\
 &= \begin{bmatrix} A_{11} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} & \cdots & A_{1n} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \\ \vdots & \ddots & \vdots \\ A_{m1} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} & \cdots & A_{mn} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \end{bmatrix} \\
 &= \begin{bmatrix} A_{11} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} & \cdots & A_{1n} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \\ \vdots & \ddots & \vdots \\ A_{m1} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} & \cdots & A_{mn} \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} A_{11} \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} & \cdots & A_{1n} \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \\ \vdots & \ddots & \vdots \\ A_{m1} \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} & \cdots & A_{mn} \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \end{bmatrix} \\
 &= \underbrace{A \otimes \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}}_{\text{well, duh, } B \oplus C = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}} + A \otimes \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \\
 &\quad \text{and use previous property just proved}
 \end{aligned}$$

However,

$$\begin{aligned}
 (A \otimes B) \oplus (A \otimes C) &= \begin{bmatrix} A \otimes B & 0 \\ 0 & A \otimes C \end{bmatrix} \quad \text{does not} \\
 &\quad \text{match the} \\
 &\quad \text{possibly nonzero} \\
 &\quad \text{blocks here in} \\
 &\quad A \otimes (B \oplus C). \\
 &\quad \underline{\text{false}}, \quad A \otimes (B \oplus C) \neq (A \otimes B) \oplus (A \otimes C).
 \end{aligned}$$

P8

$$A \otimes B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \left[\begin{array}{c|c} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & 2 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ \hline 3 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & 4 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \right] = \begin{bmatrix} 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \\ \hline 6 & 3 & 8 & 4 \\ 3 & 6 & 4 & 8 \end{bmatrix} = A \otimes B$$

$$A \oplus B = \left[\begin{array}{c|c} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\det(A) = 4 - 6 = -2$$

$$\det(B) = 4 - 1 = 3$$

$$\text{trace}(A) = 5$$

$$\text{trace}(B) = 4$$

$$\det(A \otimes B) = 36 = (-2)^2 (3)^2$$

$$\det(A \oplus B) = -6 = (-2)(3)$$

$$\text{trace}(A \otimes B) = 4 + 16 = 20 = (5)(4)$$

$$\text{trace}(A \oplus B) = 5 + 4 = 9 = 5 + 4$$

Patterns :

$$\begin{cases} \det(A \otimes B) = (\det(A))^2 (\det(B))^2 \\ \det(A \oplus B) = \det A \det B \\ \text{trace}(A \otimes B) = \text{trace}(A) \text{trace}(B) \\ \text{trace}(A \oplus B) = \text{trace}(A) + \text{trace}(B) \end{cases}$$

It is known for $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{p \times p}$ that

$$\det(A \otimes B) = (\det(A))^m \det(B)^p \quad (\text{see the Kronecker product article in Wikipedia for instance.})$$

Remark: I'm certain I could prove

these. But, I'm not certain my proofs would be worthy of the attention of others... how to give pretty proofs of the patterns above ?? Hmmm....

P9 Let $\eta(A, B) = \text{tr}(A^T B)$ $\forall A, B \in \mathbb{R}^{n \times n}$. Here $\text{tr}(M) = \sum_{i=1}^n M_{ii}$

- 1.) $\eta(A, B) = \text{tr}(A^T B)$: defⁿ of η
 $= \text{tr}((A^T B)^T)$: notice $\text{tr}(M) = \text{tr}(M^T)$
 $= \text{tr}(B^T A)$: $(A^T B)^T = B^T A^{TT} = B^T A$ prop. of transpose.
 $= \eta(B, A)$ $\therefore \underline{\eta \text{ is symmetric.}}$

- 2.) $\eta(cA + B, D) = \text{tr}((cA + B)^T D)$: defⁿ of η
 $= c\text{tr}(A^T D + B^T D)$: prop. of transpose.
 $= c\text{tr}(A^T D) + \text{tr}(B^T D)$: prop. of trace.
 $= c\eta(A, D) + \eta(B, D)$: defⁿ of η

Hence η is linear in its 1st entry. The proof $\eta(A, cB + D) = c\eta(A, B) + \eta(A, D)$ $\forall A, B, D \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}$ follows by a similar calculation or simply by symmetry of η .

- 3.) $\eta(A, A) = \text{tr}(A^T A)$: defⁿ of η
 $= \sum_{i=1}^n (A^T A)_{ii}$: defⁿ of trace
 $= \sum_{i=1}^n \sum_{j=1}^n (A^T)_{ij} A_{ji}$: defⁿ of $A^T A$ product.
 $= \sum_{i,j=1}^n (A_{ji})^2$: $(A^T)_{ij} A_{ji} = A_{ji} A_{ji} = (A_{ji})^2$.
 $= (A_{11})^2 + (A_{12})^2 + \dots + (A_{nn})^2 \geq 0$: prop. of \mathbb{R} #'s.

- 4.) $\eta(A, A) = 0 = (A_{11})^2 + \dots + (A_{nn})^2$
 shows $\eta(A, A) = 0 \iff A_{ij} = 0 \forall i, j$ hence
 $\eta(A, A) = 0 \iff A = 0$.

this completes the proof that $\eta: \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is an inner product.

P9 continued / if $\|A\| = \sqrt{\eta(A, A)}$ then,

$$\| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{1+4+9+16} = \boxed{\sqrt{30}}$$

[P10] Show $\|AB\| \leq \|A\| \cdot \|B\|$ where $\|A\| = \sqrt{\text{tr}(A^T A)}$

Notice, $\|A\|^2 = \sum_{i=1}^n \|\text{row}_i(A)\|^2 = \sum_{i=1}^n \|\text{col}_i(A)\|^2$

where $\|\text{row}_i(A)\|^2 = (A_{i1})^2 + \dots + (A_{in})^2$ & $\|\text{col}_i(A)\|^2 = (A_{1i})^2 + \dots + (A_{ni})^2$ are the usual norm on rows & columns viewed as \mathbb{R}^n with Euclidean norm. Recall for $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$

$$Ax = (\text{row}_1(A) \cdot x, \text{row}_2(A) \cdot x, \dots, \text{row}_n(A) \cdot x)$$

$$\|Ax\|^2 = \sum_{i=1}^n (\text{row}_i(A) \cdot x)^2 \leq \sum_{i=1}^n \|\text{row}_i(A)\|^2 \|x\|^2 \leftarrow \begin{array}{l} (\vec{A} \cdot \vec{x}) \leq \|\vec{A}\| \|\vec{x}\| \\ \text{Cauchy-Schwarz} \end{array}$$

$$\Rightarrow \|Ax\|^2 \leq \left(\sum_{i=1}^n \|\text{row}_i(A)\|^2 \right) \|x\|^2 = \underline{\|A\|^2 \|x\|^2} *$$

Thus, as $AB = A[B_1 | B_2 | \dots | B_n] = [AB_1 | AB_2 | \dots | AB_n]$ we have,

$$\begin{aligned} \|AB\|^2 &= \sum_{i=1}^n \|A \text{col}_i(B)\|^2 : \text{def}^* \text{ of Frob. norm and identity} \\ &\leq \sum_{i=1}^n \|A\|^2 \|\text{col}_i(B)\|^2 \leftarrow \begin{array}{l} \text{by } * \text{ with } x = \text{col}_i(B) \\ \text{applied } n\text{-fold times} \end{array} \\ &= \|A\|^2 \sum_{i=1}^n \|\text{col}_i(B)\|^2 : \text{factored constant out} \\ &\quad \text{of finite sum.} \\ &= \|A\|^2 \|B\|^2 \end{aligned}$$

$$\text{But, } \|A\| \geq 0 \Rightarrow \underline{\|AB\| \leq \|A\| \|B\|} \quad \forall A, B \in \mathbb{R}^{n \times n}.$$

This shows $(\mathbb{R}^{n \times n}, \|\cdot\|)$ forms a Banach Algebra as we already have shown $\mathbb{R}^{n \times n}$ is a complete NLS.

P11 Show $\|(x, y)\|_\infty = \max\{|x|, |y|\}$ defines
a norm on \mathbb{R}^2

$$1.) \|(c(x, y))\|_\infty = \|(cx, cy)\|_\infty = \max\{|cx|, |cy|\}$$

However, we can easily see $\max\{|cx|, |cy|\} = |c| \max\{|x|, |y|\}$

Hence $\|(c(x, y))\|_\infty = |c| \|(x, y)\|_\infty \therefore \|\cdot\|_\infty$ is absolutely homogeneous.

$$\begin{aligned} 2.) \|(x, y) + (z, w)\|_\infty &= \|(x+z, y+w)\|_\infty \\ &= \max\{|x+z|, |y+w|\} : \text{defn of } \|\cdot\|_\infty \\ &\leq \max\{|x|+|z|, |y|+|w|\} : \Delta-\text{inequality} \\ &\leq \max\{|x|, |y|\} + \max\{|z|, |w|\} : \text{for } \mathbb{R} \text{ and property of max-funct.} \\ &\quad \text{Lemma.} \\ &= \|(x, y)\|_\infty + \|(z, w)\|_\infty \end{aligned}$$

Lemma: Let $a, b, c, d \geq 0$ then $\max\{a+b, c+d\} \leq \max\{a, c\} + \max\{b, d\}$.

Proof: $\max\{a+b, c+d\} = a+b$ or $c+d$. If

$$\max\{a+b, c+d\} = a+b \leq \max\{a, c\} + \max\{b, d\}$$

If on the other hand,

$$\max\{a+b, c+d\} = c+d \leq \max\{a, c\} + \max\{b, d\}$$

Since $\max\{a, c\} \geq a, c$ & $\max\{b, d\} \geq b, d$. //

$$3.) \|(x, y)\|_\infty = 0 \Rightarrow \max\{|x|, |y|\} = 0$$

$$\Rightarrow |x| = 0 \text{ and } |y| = 0$$

$$\Rightarrow (x, y) = (0, 0)$$

Likewise $\|(0, 0)\|_\infty = 0$ hence $\|v\|_\infty = 0 \Leftrightarrow v = 0$.

$$4.) \|(x, y)\|_\infty = \max\{|x|, |y|\} \geq 0. \quad \text{In summary, } \|\cdot\|_\infty \text{ is a norm on } \mathbb{R}^2.$$

Remark: it's easy to see $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$

also defines a norm on \mathbb{R}^n . Moreover, something similar can be done for a NLS $(V, \|\cdot\|)$.

[P12] $f: U \rightarrow W$ is continuous $\iff \begin{cases} \text{inverse image of open sets} \\ \text{in } W \text{ is open in } U \end{cases}$

\Rightarrow Assume $f: U \rightarrow W$ is continuous. Let $V \subseteq W$ be an open set. If $f^{-1}(V) = \emptyset$ then \emptyset is open. Otherwise, $\exists x_0 \in f^{-1}(V) \subseteq U$. Note $x_0 \in f^{-1}(V)$
 $\Rightarrow \exists y_0 \in V$ s.t. $f(x_0) = y_0$. But, y_0 is an interior point thus $\exists \epsilon > 0$ such that $\{y \in W \mid \|y - y_0\| < \epsilon\} \subseteq V$.
But, as $\lim_{x \rightarrow x_0} f(x) = f(x_0) = y_0$ by continuity, $\exists \delta > 0$ such that $0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - y_0\| < \epsilon$. Thus $\{x \in U \mid \|x - x_0\| < \delta\}$ is an open-ball about x_0 contained in $V \therefore x_0$ is interior. Moreover, as x_0 was arbitrary we find $f^{-1}(V)$ is open.

\Leftarrow Assume the inverse image of open sets in W are open in U . Let $x_0 \in U$ and $f(x_0) \in W$. Consider the ϵ -ball about $f(x_0)$ in W ; $\{y \in W \mid \|y - f(x_0)\| < \epsilon\}$ is an open set. Thus, $f^{-1}\{y \in W \mid \|y - f(x_0)\| < \epsilon\}$ is an open set in U which contains x_0 as $y = f(x_0)$ clearly satisfies $\|f(x_0) - f(x_0)\| = 0 < \epsilon$.

Thus, $\exists \delta > 0$ for which $0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$.
Hence, $\lim_{x \rightarrow x_0} f(x) = f(x_0)$. But, $x_0 \in U$ is arbitrary $\therefore f$ is continuous on U .

P13. Let V be a real inner product space with inner-product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$. Verify $\|x\| = \sqrt{\langle x, x \rangle}$ is indeed a norm

$$\begin{aligned}
 (1.) \quad \|cx\| &= \sqrt{\langle cx, cx \rangle} && : \text{def}^{\triangleleft} \text{ of } \|\cdot\| \\
 &= \sqrt{c^2 \langle x, x \rangle} && : \text{bilinearity of } \langle \cdot, \cdot \rangle \\
 &= \sqrt{c^2} \sqrt{\langle x, x \rangle} && : \text{prop. of } \sqrt{} \\
 &= |c| \|x\| && : \text{def}^{\triangleleft} \text{ of } |\cdot| \neq \|\cdot\|.
 \end{aligned}$$

Hence $\|cx\| = |c| \|x\| \quad \forall x \in V \text{ and } c \in \mathbb{R}$.

$$(2.) \quad \|x\| = \sqrt{\langle x, x \rangle} \geq 0 \quad \text{as } \langle x, x \rangle \geq 0.$$

$$(3.) \quad \|x\| = 0 \Rightarrow \sqrt{\langle x, x \rangle} = 0 \Rightarrow \langle x, x \rangle = 0 \Leftrightarrow x = 0.$$

Likewise, $x = 0 \Rightarrow \langle x, x \rangle = 0 \Rightarrow \|x\| = 0$.

Thus, $\|x\| = 0 \Leftrightarrow x = 0$.

(4.) the triangle-inequality remains. Consider,

$$\begin{aligned}
 \|x+y\|^2 &= \langle x+y, x+y \rangle && : \text{def}^{\triangleleft} \text{ of } \|\cdot\| \\
 &= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle && : \text{bilinearity \& symmetry} \\
 &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && : \text{used C.S.} \\
 &= (\|x\| + \|y\|)^2 && : |\langle x, y \rangle| \leq \|x\|\|y\| \\
 &&& \text{proved in future homework} \quad \text{:) }
 \end{aligned}$$

$$\therefore \underline{\|x+y\| \leq \|x\| + \|y\|}$$

P14 Suppose V is v-space over \mathbb{R} with norms $\|\cdot\|_1$, $\|\cdot\|_2$ for which $\exists m, M$ such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

for all $x \in V$. Show $U \subseteq V$ is open w.r.t. $\|\cdot\|_1$ iff it is open w.r.t. $\|\cdot\|_2$

Let U be open w.r.t. $\|\cdot\|_1$. Hence, if $x_0 \in U$ then $\exists \delta > 0$ s.t. $B_\delta'(x_0) = \{x \in V \mid \|x - x_0\|_1 < \delta\} \subseteq U$.

$$\text{Notice, } \|x - x_0\|_1 < \delta \Rightarrow M\|x - x_0\|_1 \geq \|x - x_0\|_2$$

$$\text{oh, more to the point, } \|x - x_0\|_2 \leq M\|x - x_0\|_1 < \delta M$$

$$\text{thus } B_{\delta M}^2(x_0) = \{x \in V \mid \|x - x_0\|_2 < \delta M\} \subseteq B_\delta'(x_0) \subseteq U$$

thus x_0 is also interior w.r.t. $\|\cdot\|_2$ -norm. It follows U is open w.r.t. $\|\cdot\|_2$.

~~—————~~
Let U be open w.r.t. $\|\cdot\|_2$. Hence $x_0 \in U$ has

$$\delta > 0 \text{ for which } B_\delta^2(x_0) = \{x \in V \mid \|x - x_0\|_2 < \delta\} \subseteq U.$$

$$\text{However, } m\|x - x_0\|_1 \leq \|x - x_0\|_2 < \delta \Rightarrow \|x - x_0\|_1 < \delta/m$$

thus $B_{\delta/m}^1(x_0) \subseteq B_\delta^2(x_0) \subseteq U \Rightarrow x_0$ is also interior w.r.t. $\|\cdot\|_1$ -norm. It follows U is open w.r.t. $\|\cdot\|_1$.

P15) Show $\mathbb{R}^{m \times n}$ with Frob. norm is complete

Let $\{A(n)\}_{n=1}^{\infty}$ be a Cauchy sequence of matrices w.r.t.

Frobenius norm. Thus, for each $\epsilon > 0$, $\exists N \geq 1$ for which $N \leq m < n \Rightarrow \|A(n) - A(m)\| < \epsilon$. Recall,

$$A = \sum_{i,j} A_{ij} E_{ij} \text{ and } \|A\| = \sqrt{A_{11}^2 + \dots + A_{mn}^2}. \text{ Also,}$$

critical to our argument, $|A_{ij}| \leq \|A\| \quad \forall i, j$.

$$\text{Thus } |A_{ij}(n) - A_{ij}(m)| < \|A(n) - A(m)\| < \epsilon$$

which shows $\{A_{ij}(n)\}_{n=1}^{\infty}$ is Cauchy-seq in \mathbb{R} .

As \mathbb{R} is complete, $\exists B_{ij}$ s.t. $A_{ij}(n) \rightarrow B_{ij}$ as $n \rightarrow \infty$. Let $B = \sum_{i,j} B_{ij} E_{ij}$ and observe

$$\lim_{n \rightarrow \infty} (A(n)) = \sum_{i,j} \lim_{n \rightarrow \infty} (A_{ij}(n)) E_{ij} : \text{vector limit theorem!}$$

$$= \sum_{i,j} B_{ij} E_{ij}$$

$$= B \quad \therefore \{A(n)\} \text{ is convergent}$$

$$\Rightarrow \mathbb{R}^{m \times n} \text{ is complete. //}$$

P16 Show $\{A\}$ is closed set in $\mathbb{R}^{m \times n}$

We seek to show $\mathbb{R}^{m \times n} - \{A\}$ is open.

Let $X \in \mathbb{R}^{m \times n} - \{A\}$ then let $\|X - A\| = c$

and we claim $B_{c/2}(X) \subseteq \mathbb{R}^{m \times n} - \{A\}$.

It suffices to show $A \notin B_{c/2}(X)$.

Suppose towards $\rightarrow \leftarrow A \in B_{c/2}(X)$ then

$\|X - A\| < c/2$ but $\|X - A\| = c$ hence $\rightarrow \leftarrow$.

Thus $A \notin B_{c/2}(X)$ which shows X is an interior point of $\mathbb{R}^{m \times n} - \{A\}$. But as X was arbitrary $\Rightarrow \mathbb{R}^{m \times n} - \{A\}$ is open $\therefore \underline{\{A\} \text{ is closed}}$.

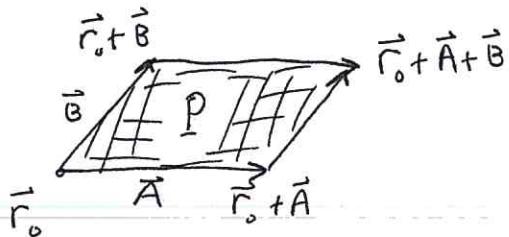
Remark: with sequential limits this is probably easier.

P17 $P = \{ \vec{r}_0 + u\vec{A} + v\vec{B} \mid (u, v) \in [0, 1]^2 \}$

$$\vec{A} = (\vec{A} \cdot \hat{u})\hat{u} + \hat{u} \times (\hat{u} \times \vec{A})$$

fun fact from
Math 231 FALL 2015

(a.) find area of P ; $\text{Area}(P)$



$$\text{Area}(P) = \|\vec{A} \times \vec{B}\|.$$

(b.) $L_{ij}(\vec{v}) = (\vec{v} \cdot \hat{x}_i)\hat{x}_i + (\vec{v} \cdot \hat{x}_j)\hat{x}_j$ is a linear transformation since the dot-product has the needed linearity:

$$\begin{aligned} L_{ij}(c\vec{v} + \vec{w}) &= [(c\vec{v} + \vec{w}) \cdot \hat{x}_i]\hat{x}_i + [(c\vec{v} + \vec{w}) \cdot \hat{x}_j]\hat{x}_j \\ &= c[(\vec{v} \cdot \hat{x}_i)\hat{x}_i + (\vec{v} \cdot \hat{x}_j)\hat{x}_j] + (\vec{w} \cdot \hat{x}_i)\hat{x}_i + (\vec{w} \cdot \hat{x}_j)\hat{x}_j \\ &= c L_{ij}(\vec{v}) + L_{ij}(\vec{w}). \end{aligned}$$

(c.) $L_{ij}(P) = \{ L_{ij}(\vec{r}_0) + u L_{ij}(\vec{A}) + v L_{ij}(\vec{B}) \mid (u, v) \in [0, 1]^2 \}$

$$\text{Area}(L_{ij}(P)) = \|L_{ij}(\vec{A}) \times L_{ij}(\vec{B})\|$$

Notice $\vec{A} = (\vec{A} \cdot \hat{x}_1)\hat{x}_1 + (\vec{A} \cdot \hat{x}_2)\hat{x}_2 + (\vec{A} \cdot \hat{x}_3)\hat{x}_3$

thus $L_{ij}(\vec{A}) = \vec{A} - (\vec{A} \cdot \hat{x}_h)\hat{x}_h$ where $\{i, j, h\} = \{1, 2, 3\}$.

But, $\vec{A} - (\vec{A} \cdot \hat{u})\hat{u} = \hat{u} \times (\hat{u} \times \vec{A})$ thus $L_{ij}(\vec{A}) = \hat{x}_h \times (\hat{x}_h \times \vec{A})$ where $i, j \neq h$.

$$\text{Area}(L_{ij}(P)) = \|\hat{x}_h \times (\hat{x}_h \times \vec{A}) \times \hat{x}_h \times (\hat{x}_h \times \vec{B})\|$$

Ok, fine, this didn't help. I'll try something else ↴

P17 continued

$$\begin{aligned}\text{Area}(L_{ij}(P)) &= \| L_{ij}(\vec{A}) \times L_{ij}(\vec{B}) \| \\&= \| [(\hat{x}_i \cdot \vec{A}) \hat{x}_i + (\hat{x}_j \cdot \vec{A}) \hat{x}_j] \times [(\hat{x}_i \cdot \vec{B}) \hat{x}_i + (\hat{x}_j \cdot \vec{B}) \hat{x}_j] \| \\&= \| (\hat{x}_i \cdot \vec{A})(\hat{x}_j \cdot \vec{B}) \hat{x}_i \times \hat{x}_j + (\hat{x}_j \cdot \vec{A})(\hat{x}_i \cdot \vec{B}) \hat{x}_j \times \hat{x}_i \| \\&= \| (A_i B_j - A_j B_i) \hat{x}_i \times \hat{x}_j \| \\&= \| (\vec{A} \times \vec{B}) \cdot (\hat{x}_i \times \hat{x}_j) \| \end{aligned}$$

Thus,

$$\| \vec{A} \times \vec{B} \|^2 = (\vec{A} \times \vec{B})_1^2 + (\vec{A} \times \vec{B})_2^2 + (\vec{A} \times \vec{B})_3^2$$

$$\text{Area}(P)^2 = \text{Area}(L_{23}(P))^2 + \text{Area}(L_{31}(P))^2 + \text{Area}(L_{12}(P))^2$$

Remark: there is a generalization of this

to n -dimensions based on the $(n-1)$ -ary product $\vec{V}_1 \times \vec{V}_2 \times \dots \times \vec{V}_{n-1} = \det \left[\begin{array}{c|c|c|c|c} \hat{x}_1 & \hat{x}_2 & \dots & \hat{x}_n \\ \hline \vec{v}_1 & & & \\ \hline & \vdots & & \\ \hline & & & \vec{v}_n \end{array} \right]$

We also have for orthonormal $\vec{v}_{i_1}, \dots, \vec{v}_{i_{n-1}}$,

$$\vec{A} \cdot (\vec{v}_{i_1} \times \vec{v}_{i_2} \times \dots \times \vec{v}_{i_{n-1}}) = \vec{A} \cdot \vec{v}_k$$

where $i_1, i_2, \dots, i_{n-1} \neq k$. And...

(see Munkres ANALYSIS ON MANIFOLDS For
a nice discussion on det. & volume in general.)

[P18]

$$A_3 = \left\{ A \in \mathbb{R}^{3 \times 3} \mid A^T = -A \right\} = \text{span} \left\{ \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{f_1}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{f_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{f_3} \right\}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = - \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$f_1 = E_{12} - E_{21}$$

$$f_2 = E_{13} - E_{31}$$

$$f_3 = E_{23} - E_{32}$$

$$a = -a, \quad b = -e, \quad i = -i \Rightarrow a = b = i = 0$$

$$\left. \begin{array}{l} b = -d, \quad c = -g, \quad h = -f \\ \beta = \{E_{12} - E_{21}, \quad E_{13} - E_{31}, \quad E_{23} - E_{32}\} \end{array} \right\}$$

Thus $\beta = \{E_{12} - E_{21}, \quad E_{13} - E_{31}, \quad E_{23} - E_{32}\}$ as above. And,

$$\Phi_p \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = (a, b, c)$$

Consider,

$$A\Sigma = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix} = \begin{bmatrix} -ax - by & -bz & az \\ -cy & -ax - cz & -ay \\ cx & -bx & -by - cz \end{bmatrix}$$

$$\Sigma A = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} -ax - by & -cy & cx \\ -bz & -ax - cz & -bx \\ az & -ay & -by \end{bmatrix}$$

$$A\Sigma - \Sigma A = [A, \Sigma] = \begin{bmatrix} 0 & -bz + cy & az - cx \\ -(-bz + cy) & 0 & bx - ay \\ -(az - cx) & -(bx - ay) & 0 \end{bmatrix}$$

$$\text{But, } \langle a, b, c \rangle \times \langle x, y, z \rangle = \langle bz - cy, cx - az, ay - bx \rangle$$

$$\text{Thus, } [\Phi_p^{-1}(a, b, c), \Phi_p^{-1}(x, y, z)] = \Phi_p^{-1}((x, y, z) \times (a, b, c))$$

$$\text{or, } [\Phi_p^{-1}(v), \Phi_p^{-1}(w)] = \Phi_p^{-1}(w \times v)$$

The commutator reversed gives the cross-product.