Same instructions as Mission 1. Thanks!

Problem 19 Your signature below indicates you have:

(a.) I have read much of Cook's Chapter 4 and the Chapter 3 update:________________.

Problem 20 Find the Jacobian matrix for $F(x, y) = (x^2 - y^2, 2xy)$. Use the Jacobian matrix to help construct a linearization of F centered at $(1, 1)$.

Problem 21 Find the Jacobian matrix for $G(x, y, z) = x + y^2 + z^3$. Use the Jacobian matrix to help construct a linearization of F centered at $(1, \sqrt{2}, \sqrt[3]{3})$.

Problem 22 Find the Jacobian matrix for $H(r, t, s) = (r\cos(t)\sinh(s), r\cos(t)\sinh(s), r\cosh(s))$.

Problem 23 Let V, W be finite-dimensional normed linear spaces. Show that if $F : V \to W$ is Frechet differentiable at $a \in V$ then F is continuous at a.
Hint: following Edward's advice, define R_F by:

$$R_F = \frac{F(a+h) - F(a) - d_aF(h)}{||h||}$$

and observe $R_F \to 0$ as $h \to 0$ if F is differentiable at $a \in V$. Solve for $F(a + h)$ and show that $\lim_{h\to 0} F(a + h) = F(a)$. You may use limit laws here, you should not need an ϵ, δ type argument.

Problem 24 Suppose $\beta = \{w_1, \dots, w_m\}$ and $\bar{\beta} = \{\overline{w}_1, \dots, \overline{w}_m\}$ are bases for a normed linear space W. Furthermore, suppose V is a normed linear space and $F : V \to W$ is a function with component functions F_i with respect to β and $\overline{F}_i$ with respect to $\overline{\beta}$. That is:

$$F = \sum F_i w_i \qquad \& \qquad F = \sum \overline{F}_i \overline{w}_i.$$

Let $B \in W$ where $B = \sum_i B_i w_i$ and $B = \sum_i \overline{B}_i \overline{w}_i$. Given this set-up:
Prove: if $\lim_{x\to a} F_i(x) = B_i$ for $i = 1, \dots, m$ then $\lim_{x\to a} \overline{F}_i(x) = \overline{B}_i$ for $i = 1, \dots, m$.

Problem 25 Suppose $x_1, \dots, x_n$ are coordinates of a normed linear space V with respect to the basis $\beta = \{v_1, \dots, v_n\}$. Let $F, G : V \to \mathbb{R}$ be differentiable functions on V and $h : \mathbb{R} \to \mathbb{R}$ a differentiable function on $\mathbb{R}$. Show: for $c \in \mathbb{R}$ and for $i = 1, \dots, n$,

$$\frac{\partial}{\partial x_i}\left[cF(x) + G(x)\right] = c\frac{\partial F}{\partial x_i} + \frac{\partial G}{\partial x_i} \qquad \& \qquad \frac{\partial}{\partial x_i}\left[h(F(x))\right] = h'(F(x))\frac{\partial F}{\partial x_i}.$$

Problem 26 Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $F(A) = A^3$. Calculate the Frechet differential $dF_A(H)$ in two ways:

(a.) calculate $F(A + H) - F(A)$ and select what appears linear in H and claim it is $dF_A(H)$, show this is correct by working through the Frechet limit directly (with a squeeze theorem argument much as we did for the A^2 in the 2015 Lecture 3).

(b.) let X_{ij} be coordinates of $\mathbb{R}^{n\times n}$ with respect to the usual basis of unit-matrices $\{E_{ij}\}$, calculate partial derivatives of F, observe the formulas are all clearly continuous, hence construct $d_AF(H)$ by piecing together the partial derivatives. (we also did this for the A^2 function in the 2015 Lecture 3)

Logically, the following is superfluous as we proved a product rule in Lecture 4 which includes the result of the problem which follows as well as many others, but, I still think the problem is worthwhile so I include it. After all, the problems are for you to learn.

Problem 27 Show that if $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and $G : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ are differentiable at A then the matrix product function FG is likewise differentiable at A.
Hint: use the technique introduced in Problem 23

Problem 28 Use chain-rule for $f(x, y) = \sqrt[y]{x}$ composed with $\gamma(t) = (t, t)$ to calculate $\frac{d}{dt}[f(\gamma(t))]$. *Thus, in view of the fact* $\sqrt[t]{t} = f(\gamma(t))$ *you have calculated* $\frac{d}{dt}\left[\sqrt[t]{t}\right]$.

Problem 29 Suppose $X : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $X(s, t) = (x(s, t), y(s, t), z(s, t))$ and $\bar{X} : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $\bar{X}(\bar{s}, \bar{t}) = (\bar{x}(\bar{s}, \bar{t}), \bar{y}(\bar{s}, \bar{t}), \bar{z}(\bar{s}, \bar{t}))$. Suppose further that there exists some notation changing map $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $X = \bar{X} \circ T$ meaning:

$$\bar{X}(T(s, t)) = X(s, t)$$

where the notation $T(s, t) = (\bar{s}(s, t), \bar{t}(s, t))$ yields

$$\bar{X}((\bar{s}(s, t), \bar{t}(s, t))) = X(s, t)$$

Find $\frac{\partial X}{\partial s}$ and $\frac{\partial X}{\partial t}$ in terms of $\frac{\partial \bar{X}}{\partial \bar{s}}$, $\frac{\partial \bar{X}}{\partial \bar{t}}$ and $\frac{\partial \bar{s}}{\partial s}$, $\frac{\partial \bar{s}}{\partial t}$ and $\frac{\partial \bar{t}}{\partial s}$, $\frac{\partial \bar{t}}{\partial t}$.

Problem 30 (this is partly a continuation of the previous problem) Recall the surface integral of a vector field $\vec{F}$ on a surface S parameterized by $X : D \to S$ was defined by $\int_S \vec{F} \bullet d\vec{S} = \iint_D \vec{F}(X(s, t)) \bullet \left(\frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t}\right) ds\, dt$. Show that this definition is independent of the choice of parametrization. In particular, show that if you replace the expressions in terms of X and s, t in terms of $\bar{X}$ then you obtain the surface integral written in terms of the barred-parametrization. However, this is only true if we impose a certain condition on T. What condition is that?

Problem 31 Show that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is invertible on $U \subseteq \mathbb{R}^n$ then dF_{x_o} is invertible for each $x_o \in U$.

Problem 32 Edwards #3.11 on page 89.

Problem 33 Problem 1.36 on page 23 of the handout from Renteln's *Manifolds, Tensors, and Forms: an Introduction for Mathematicians and Physicists* (you should read 1.32 and use it)

Problem 34 Problem 1.46 on page 25-26 of the handout from Renteln's *Manifolds, Tensors, and Forms: an Introduction for Mathematicians and Physicists* (this problem makes you smarter)

Problem 35 Problem 1.49 on page 27 of the handout from Renteln's *Manifolds, Tensors, and Forms: an Introduction for Mathematicians and Physicists* (this problem is not that bad)

Problem 36 Let V be a real vector space with norm $||\cdot||$. The purpose of this problem is to establish the following equivalence: the norm is induced from an inner product $\Leftrightarrow$ the norm satisfies the parallelogram law below:

$$\boxed{||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)} \quad \star$$

for all $x, y \in V$. The proof is somewhat involved:

(a.) Suppose there exists an inner product $\langle , \rangle : V \times V \to \mathbb{R}$ for which $||x|| = \sqrt{\langle x, x\rangle}$ for all $x \in V$. Show $||\cdot||$ so-defined satisfies the parallelogram law:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

for all $x, y \in V$.
(this proves the $\Rightarrow$ of the claim, the rest of the problem goes to the other direction)

(b.) Suppose there exists an inner product $\langle , \rangle : V \times V \to \mathbb{R}$ for which $||x|| = \sqrt{\langle x, x\rangle}$ for all $x \in V$. Show $||\cdot||$ so-defined satisfies derive the the *polar form identity*:

$$\langle x, y\rangle = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$$

(c.) Assume V is a given real normed linear space with norm $||\cdot||$ **which satisfies the identity** $\star$. In view of the result of the previous part, it is natural to define $g : V \times V \to \mathbb{R}$ by the following formula:

$$g(x, y) = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$$

as a potential inner-product induced from the given norm.

- **(i.)** show $g(x, y) = g(y, x)$,
- **(ii.)** show $g(x, x) = ||x||^2$ and hence explain why g is positive definite,
- **(iii.)** show $g(x+y, z) = g(x, z) + g(y, z)$. (be sure to implement $\star$!)
- **(iv.)** show $g(kx, y) = kg(x, y)$ for all $k \in \mathbb{N}$ by induction on k,
- **(v.)** show $g(-x, y) = -g(x, y)$ and show $g(zx, y) = zg(x, y)$ for all $z \in \mathbb{Z}$,
- **(vi.)** show $g\left(\frac{p}{q}x, y\right) = \frac{p}{q}g(x, y)$ for all $p, q \in \mathbb{Z}$ with $q \neq 0$
- **(vii.)** Fix $y \in V$ and define $h(x) = g(x, y)$. Show $h : V \to \mathbb{R}$ is continuous on V,
- **(viii.)** let $r \in \mathbb{R}$ then there exists a sequence of rational numbers p_n/q_n converging to r as $n \to \infty$ by the density of the rational numbers in $\mathbb{R}$. Use the equivalence of sequential limits and topological $(\epsilon - \delta)$ limits paired with the continuity of h (see part vii.) to show $g(rx, y) = rg(x, y)$ for all $r \in \mathbb{R}$.
- **(ix.)** show $g(x, ry + z) = rg(x, y) + g(x, z)$ for all $x, y, z \in V$ and $r \in \mathbb{R}$. *hint: use i., iii. and x.* .

Thus we have shown g so-defined is a symmetric, positive definite, bilinear form on V which means g defines an inner-product. *This completes the $\Leftarrow$ part of the claim.*