

Same instructions as Mission 1. Thanks!

**Problem 19** Your signature below indicates you have:

(a.) I have read much of Cook's Chapter 4 and the Chapter 3 update: \_\_\_\_\_.

**Problem 20** Find the Jacobian matrix for  $F(x, y) = (x^2 - y^2, 2xy)$ . Use the Jacobian matrix to help construct a linearization of  $F$  centered at  $(1, 1)$ .

**Problem 21** Find the Jacobian matrix for  $G(x, y, z) = x + y^2 + z^3$ . Use the Jacobian matrix to help construct a linearization of  $G$  centered at  $(1, \sqrt{2}, \sqrt[3]{3})$ .

**Problem 22** Find the Jacobian matrix for  $H(r, t, s) = (r \cos(t) \sinh(s), r \overset{\sin}{\cos}(t) \sinh(s), r \cosh(s))$ .

**Problem 23** Let  $V, W$  be finite-dimensional normed linear spaces. Show that if  $F : V \rightarrow W$  is Frechet differentiable at  $a \in V$  then  $F$  is continuous at  $a$ .

*Hint: following Edward's advice, define  $R_F$  by:*

$$R_F = \frac{F(a+h) - F(a) - d_a F(h)}{\|h\|}$$

*and observe  $R_F \rightarrow 0$  as  $h \rightarrow 0$  if  $F$  is differentiable at  $a \in V$ . Solve for  $F(a+h)$  and show that  $\lim_{h \rightarrow 0} F(a+h) = F(a)$ . You may use limit laws here, you should not need an  $\epsilon, \delta$  type argument.*

**Problem 24** Suppose  $\beta = \{w_1, \dots, w_m\}$  and  $\bar{\beta} = \{\bar{w}_1, \dots, \bar{w}_m\}$  are bases for a normed linear space  $W$ . Furthermore, suppose  $V$  is a normed linear space and  $F : V \rightarrow W$  is a function with component functions  $F_i$  with respect to  $\beta$  and  $\bar{F}_i$  with respect to  $\bar{\beta}$ . That is:

$$F = \sum F_i w_i \quad \& \quad F = \sum \bar{F}_i \bar{w}_i.$$

Let  $B \in W$  where  $B = \sum_i B_i w_i$  and  $B = \sum_i \bar{B}_i \bar{w}_i$ . Given this set-up:

**Prove:** if  $\lim_{x \rightarrow a} F_j(x) = B_j$  for  $j = 1, \dots, m$  then  $\lim_{x \rightarrow a} \bar{F}_i(x) = \bar{B}_i$  for  $i = 1, \dots, m$ .

**Problem 25** Suppose  $x_1, \dots, x_n$  are coordinates of a normed linear space  $V$  with respect to the basis  $\beta = \{v_1, \dots, v_n\}$ . Let  $F, G : V \rightarrow \mathbb{R}$  be differentiable functions on  $V$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  a differentiable function on  $\mathbb{R}$ . Show: for  $c \in \mathbb{R}$  and for  $i = 1, \dots, n$ ,

$$\frac{\partial}{\partial x_i} [cF(x) + G(x)] = c \frac{\partial F}{\partial x_i} + \frac{\partial G}{\partial x_i} \quad \& \quad \frac{\partial}{\partial x_i} [h(F(x))] = h'(F(x)) \frac{\partial F}{\partial x_i}.$$

**Problem 26** Let  $F : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  be defined by  $F(A) = A^3$ . Calculate the Frechet differential  $dF_A(H)$  in two ways:

(a.) calculate  $F(A + H) - F(A)$  and select what appears linear in  $H$  and claim it is  $dF_A(H)$ , show this is correct by working through the Frechet limit directly (with a squeeze theorem argument much as we did for the  $A^2$  in the 2015 Lecture 3).

- (b.) let  $X_{ij}$  be coordinates of  $\mathbb{R}^{n \times n}$  with respect to the usual basis of unit-matrices  $\{E_{ij}\}$ , calculate partial derivatives of  $F$ , observe the formulas are all clearly continuous, hence construct  $d_A F(H)$  by piecing together the partial derivatives. (we also did this for the  $A^2$  function in the 2015 Lecture 3)

*Logically, the following is superfluous as we proved a product rule in Lecture 4 which includes the result of the problem which follows as well as many others, but, I still think the problem is worthwhile so I include it. After all, the problems are for you to learn.*

**Problem 27** Show that if  $F : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  and  $G : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  are differentiable at  $A$  then the matrix product function  $FG$  is likewise differentiable at  $A$ .

*Hint: use the technique introduced in Problem 23*

**Problem 28** Use chain-rule for  $f(x, y) = \sqrt[n]{x}$  composed with  $\gamma(t) = (t, t)$  to calculate  $\frac{d}{dt} [f(\gamma(t))]$ .  
Thus, in view of the fact  $\sqrt[n]{t} = f(\gamma(t))$  you have calculated  $\frac{d}{dt} [\sqrt[n]{t}]$ .

**Problem 29** Suppose  $X : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is given by  $X(s, t) = (x(s, t), y(s, t), z(s, t))$  and  $\bar{X} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is given by  $\bar{X}(\bar{s}, \bar{t}) = (\bar{x}(\bar{s}, \bar{t}), \bar{y}(\bar{s}, \bar{t}), \bar{z}(\bar{s}, \bar{t}))$ . Suppose further that there exists some notation changing map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $X = \bar{X} \circ T$  meaning:

$$\bar{X}(T(s, t)) = X(s, t)$$

where the notation  $T(s, t) = (\bar{s}(s, t), \bar{t}(s, t))$  yields

$$\bar{X}((\bar{s}(s, t), \bar{t}(s, t))) = X(s, t)$$

Find  $\frac{\partial X}{\partial s}$  and  $\frac{\partial X}{\partial t}$  in terms of  $\frac{\partial \bar{X}}{\partial \bar{s}}$ ,  $\frac{\partial \bar{X}}{\partial \bar{t}}$  and  $\frac{\partial \bar{s}}{\partial s}$ ,  $\frac{\partial \bar{s}}{\partial t}$  and  $\frac{\partial \bar{t}}{\partial s}$ ,  $\frac{\partial \bar{t}}{\partial t}$ .

**Problem 30** (this is partly a continuation of the previous problem) Recall the surface integral of a vector field  $\vec{F}$  on a surface  $S$  parameterized by  $X : D \rightarrow S$  was defined by  $\int_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(X(s, t)) \cdot \left( \frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t} \right) ds dt$ . Show that this definition is independent of the choice of parametrization. In particular, show that if you replace the expressions in terms of  $X$  and  $s, t$  in terms of  $\bar{X}$  then you obtain the surface integral written in terms of the barred-parametrization. However, this is only true if we impose a certain condition on  $T$ . What condition is that?

**Problem 31** Show that if  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is invertible on  $U \subseteq \mathbb{R}^n$  then  $dF_{x_o}$  is invertible for each  $x_o \in U$ .

**Problem 32** Edwards #3.11 on page 89.

**Problem 33** Problem 1.36 on page 23 of the handout from Renteln's *Manifolds, Tensors, and Forms: an Introduction for Mathematicians and Physicists* (you should read 1.32 and use it)

**Problem 34** Problem 1.46 on page 25-26 of the handout from Renteln's *Manifolds, Tensors, and Forms: an Introduction for Mathematicians and Physicists* (this problem makes you smarter)

**Problem 35** Problem 1.49 on page 27 of the handout from Renteln's *Manifolds, Tensors, and Forms: an Introduction for Mathematicians and Physicists* (this problem is not that bad)

**Problem 36** parallelogram identity

**Problem 36** Let  $V$  be a real vector space with norm  $\|\cdot\|$ . The purpose of this problem is to establish the following equivalence: the norm is induced from an inner product  $\Leftrightarrow$  the norm satisfies the parallelogram law below:

$$\boxed{\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)} \quad *$$

for all  $x, y \in V$ . The proof is somewhat involved:

- (a.) Suppose there exists an inner product  $\langle , \rangle : V \times V \rightarrow \mathbb{R}$  for which  $\|x\| = \sqrt{\langle x, x \rangle}$  for all  $x \in V$ . Show  $\|\cdot\|$  so-defined satisfies the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all  $x, y \in V$ .

(this proves the  $\Rightarrow$  of the claim, the rest of the problem goes to the other direction)

- (b.) Suppose there exists an inner product  $\langle , \rangle : V \times V \rightarrow \mathbb{R}$  for which  $\|x\| = \sqrt{\langle x, x \rangle}$  for all  $x \in V$ . Show  $\|\cdot\|$  so-defined satisfies derive the the *polar form identity*:

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2)$$

- (c.) Assume  $V$  is a given real normed linear space with norm  $\|\cdot\|$  which satisfies the **identity** \*. In view of the result of the previous part, it is natural to define  $g : V \times V \rightarrow \mathbb{R}$  by the following formula:

$$g(x, y) = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2)$$

as a potential inner-product induced from the given norm.

- (i.) show  $g(x, y) = g(y, x)$ ,
- (ii.) show  $g(x, x) = \|x\|^2$  and hence explain why  $g$  is positive definite,
- (iii.) show  $g(x + y, z) = g(x, z) + g(y, z)$ . (be sure to implement \* !)
- (iv.) show  $g(kx, y) = kg(x, y)$  for all  $k \in \mathbb{N}$  by induction on  $k$ ,
- (v.) show  $g(-x, y) = -g(x, y)$  and show  $g(zx, y) = zg(x, y)$  for all  $z \in \mathbb{Z}$ ,
- (vi.) show  $g\left(\frac{p}{q}x, y\right) = \frac{p}{q}g(x, y)$  for all  $p, q \in \mathbb{Z}$  with  $q \neq 0$
- (vii.) Fix  $y \in V$  and define  $h(x) = g(x, y)$ . Show  $h : V \rightarrow \mathbb{R}$  is continuous on  $V$ ,
- (viii.) let  $r \in \mathbb{R}$  then there exists a sequence of rational numbers  $p_n/q_n$  converging to  $r$  as  $n \rightarrow \infty$  by the density of the rational numbers in  $\mathbb{R}$ . Use the equivalence of sequential limits and topological  $(\epsilon - \delta)$  limits paired with the continuity of  $h$  (see part vii.) to show  $g(rx, y) = rg(x, y)$  for all  $r \in \mathbb{R}$ .
- (ix.) show  $g(x, ry + z) = rg(x, y) + g(x, z)$  for all  $x, y, z \in V$  and  $r \in \mathbb{R}$ . hint: use i., iii. and x. .

Thus we have shown  $g$  so-defined is a symmetric, positive definite, bilinear form on  $V$  which means  $g$  defines an inner-product. This completes the  $\Leftarrow$  part of the claim.

# MATH 332 : SOLUTION TO MISSION 2

[P20]  $F(x, y) = (x^2 - y^2, 2xy)$ . Find  $J_F$  and construct a linearization of  $F$  centered at  $(1,1)$

$$J_F = \left[ \begin{array}{c|c} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \hline 2x & -2y \\ 2y & 2x \end{array} \right] *$$

$$L_F(x, y) = F(1, 1) + J_F(1, 1) \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = (0, 2) + \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$L(x, y) = (2(x-1) - 2(y-1), 2 + 2(x-1) + 2(y-1))$$

Remark: for those taking complex analysis,

$$F \leftrightarrow f(z) = z^2 \text{ and } (1, 1) \leftrightarrow 1+i \text{ so } f'(1+i) = 2z|_{1+i} = 2(1+i)$$

$$\text{and } L_f(z) = f(1+i) + f'(1+i)(z-1-i)$$

$$L_f(z) = (1+i)^2 + 2(1+i)(z-1-i) - 2i$$

$$L_f(z) = 2i + 2(1+i)x + 2(1+i)i y + 2(1+i)(-1-i)$$

$$L_f(z) = 2i + 2x + 2ix + 2yi - 2y - 4i$$

$$L_f(z) = 2(x-y) + i(2(x+y) - 2)$$

compare to \*\*  
it's the same under  
 $1 = e_1, i = e_2$ .

[P21]  $G(x, y, z) = x + y^2 + z^3$ . Linearize  $G$  at

$$(1, \sqrt{2}, \sqrt[3]{3}) \text{ where } G(1, \sqrt{2}, \sqrt[3]{3}) = 1 + 2 + 3 = 6.$$

$$J_G(x, y, z) = [\partial_x G, \partial_y G, \partial_z G] = [1, 2y, 3z^2] = G'(x, y, z)$$

$$L(x, y, z) = 6 + G'(1, \sqrt{2}, \sqrt[3]{3}) \begin{bmatrix} x-1 \\ y-\sqrt{2} \\ z-\sqrt[3]{3} \end{bmatrix} \quad G'(1, \sqrt{2}, \sqrt[3]{3}) = [1, 2\sqrt{2}, 3 \cdot \sqrt[3]{3}]$$

$$L(x, y, z) = 6 + 1(x-1) + 2\sqrt{2}(y-\sqrt{2}) + 3\sqrt[3]{3}(z-\sqrt[3]{3})$$

P22 Find the Jacobian matrix for  $H(r, t, s) = (r \cos t \sinh s, r \sin t \sinh s, r \cosh s)$

$$H'(r, s, t) = \left[ \frac{\partial H}{\partial r} \mid \frac{\partial H}{\partial s} \mid \frac{\partial H}{\partial t} \right] = \begin{bmatrix} \cos t \sinh s & r \cos t \cosh s & -r \sin t \sinh s \\ \sin t \sinh s & r \sin t \cosh s & r \cos t \sinh s \\ \cosh s & r \sinh s & 0 \end{bmatrix}$$

$\hat{=}$  (oops, not what I asked for)  $\hat{=}$  (your answer has columns 2  $\leftrightarrow$  3.)

P23 Let  $V, W$  be finite-dim'l normed linear spaces. Show that  $F: V \rightarrow W$  Frechet diff. at  $a \in V \Rightarrow F$  continuous at  $a$ .

Define, for  $h \neq 0$ ,  $R_F(h) = \frac{F(ath) - F(a) - d_a F(h)}{\|h\|}$

Observe,  $\lim_{h \rightarrow 0} (R_F(h)) = 0$  is given by def<sup>n</sup> of diff. of  $F$  at  $a$ .

Thus, as  $F(ath) = F(a) + d_a F(h) + R_F(h)\|h\|$ ,

$$\begin{aligned} \lim_{h \rightarrow 0} [F(ath)] &= \lim_{h \rightarrow 0} (F(a) + d_a F(h) + R_F(h)\|h\|) \\ &= \lim_{h \rightarrow 0} (F(a)) + \lim_{h \rightarrow 0} (d_a F(h)) + \lim_{h \rightarrow 0} (R_F(h)) \lim_{h \rightarrow 0} \|h\| \\ &= F(a) + d_a F(0) + 0 \cdot 0 \quad : \text{by Lemma.} \\ &= \underline{F(a)}. \end{aligned}$$

Lemma:  $d_a F: V \rightarrow W$  is continuous on  $V$

Proof: for  $\beta = \{v_1, \dots, v_n\}$  a basis for  $V$  and  $\gamma = \{w_1, \dots, w_m\}$  a basis for  $W$  then  $L: V \rightarrow W$  a linear transformation may be expressed as  $L(\sum_{i=1}^n x_i v_i) = \sum_{i,j} A_{ij} x_j w_i$ . If we assume  $\beta, \gamma$  are unit-length with  $|x_j| \leq \|x\| \not\equiv |y_i| \leq \|y\|$  for each  $x \in V$  and  $y \in W$  then let  $M = \max \{|A_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}$  hence  $\|L(x)\| = \left\| \sum_{i=1}^m \sum_{j=1}^n A_{ij} x_j w_i \right\| \leq \sum_{i=1}^m \sum_{j=1}^n |A_{ij}| |x_j| \|w_i\|^2 \leq mn M N$  where  $N = \max \{|x_j| \mid j=1, 2, \dots, n\}$

continued



proof of Lemma: linear frct is continuous

Let  $\epsilon > 0$ , choose  $\delta = \frac{\epsilon}{mnM}$  where  $M = \max \{|A_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ .

Suppose  $x \in V$  where  $0 \leq \|x - x_0\| < \delta$ . Consider,  $L: V \rightarrow W$  linear  
(as on previous pg.).

$$\|L(x) - L(x_0)\| = \|L(x - x_0)\| \quad : L \text{ linear}$$

$$= \left\| \sum_{i=1}^m \sum_{j=1}^n A_{ij} (x - x_0)_j w_i \right\| : \begin{matrix} \text{assuming } A_{ij} \\ \text{is matrix of } L \\ \text{w.r.t. } \beta, \gamma \end{matrix}$$

$$\leq \sum_{i=1}^m \sum_{j=1}^n |A_{ij}| |(x - x_0)_j| \quad \overbrace{\|w_i\|}^{\text{normalized.}}$$

$$\leq \sum_{i=1}^m \sum_{j=1}^n M \|x - x_0\| \quad : \begin{matrix} \text{as we assume } \beta \text{ is} \\ \text{basis with} \\ |(x - x_0)_j| \leq \|x - x_0\|. \end{matrix}$$

$$< M \delta \sum_{i=1}^m \sum_{j=1}^n 1.$$

$$= M \frac{\epsilon}{mnM} mn$$

$$= \epsilon.$$

Thus  $\lim_{x \rightarrow x_0} L(x) = L(x_0)$  and as  $x_0 \in V$  was arbitrary  
we have continuity of  $L$  on  $V$ . Finally, note  
 $d_a F: V \rightarrow W$  is a linear transformation thus  
 $d_a F$  is continuous on  $V$ .

Remark:  $a \in V$  is fixed in the comment above. If  
we seek to describe continuity of  $a \mapsto d_a F$   
then further theory is needed (it's not finite-dim'l  
vector space continuity, rather, that of  $\infty$ -dim'l frct-spaces...)

Remark: [P24] shows there is no loss of generality in  
assuming normality and  $|x_j| \leq \|x\|$  etc.

[P24] Let  $\beta = \{w_1, \dots, w_m\}$  and  $\bar{\beta} = \{\bar{w}_1, \dots, \bar{w}_m\}$  be bases for NLS  $W$ .

Also, let  $V$  be a NLS with  $F: V \rightarrow W$  a function with component functions  $F_i$  with respect to  $\beta$  and  $\bar{F}_i$  with respect to  $\bar{\beta}$ ,

$$F = \sum_i F_i w_i \quad \text{and} \quad \bar{F} = \sum_i \bar{F}_i \bar{w}_i;$$

Let  $B \in W$  and  $B = \sum_i B_i w_i = \sum_i \bar{B}_i \bar{w}_i$ . Given this

set-up prove: if  $\lim_{x \rightarrow a} F_j(x) = B_j$  for  $j=1, 2, \dots, m$  then  $\lim_{x \rightarrow a} \bar{F}_j(x) = \bar{B}_j$  for  $j=1, 2, \dots, m$ .

Suppose  $\lim_{x \rightarrow a} F_j(x) = B_j$  for  $j=1, 2, \dots, m$ . Since  $\text{span } \bar{\beta} = W$

and  $w_i \in W$  for each  $i=1, 2, \dots, m$ , it follows  $\exists P_{ij} \in \mathbb{R}$  for

which  $w_i = \sum_j P_{ij} \bar{w}_j$ . Furthermore, comparing  $\beta, \bar{\beta}$  expansions of  $F$ ,

$$F = \sum_i F_i w_i = \sum_{i,j=1}^m F_i P_{ij} \bar{w}_j = \sum_{j=1}^m \bar{F}_j \bar{w}_j$$

we derive from LI of  $\{\bar{w}_1, \dots, \bar{w}_n\}$  that

$$\bar{F}_j = \sum_{i=1}^m F_i P_{ij}$$

Likewise,  $\bar{B}_j = \sum_{i=1}^m B_i P_{ij}$ . Consider,

$$\begin{aligned} \lim_{x \rightarrow a} \bar{F}_j(x) &= \lim_{x \rightarrow a} \left( \sum_{i=1}^m F_i(x) P_{ij} \right) \\ &= \sum_{i=1}^m P_{ij} \left( \lim_{x \rightarrow a} F_i(x) \right) \\ &= \sum_{i=1}^m P_{ij} B_i \\ &= \bar{B}_j. \end{aligned}$$

using linearity  
of limit paired  
with the given  
assumption \*

Remark: Since  $V = \text{span } \{v_1, \dots, v_n\}$  can be given  $\langle v_i, v_j \rangle = \delta_{ij}$

and  $\|x\| = \sqrt{\langle x, x \rangle}$  it follows  $\exists \beta$  for  $V$  s.t.  $|x_i| \leq \|x\|$

where  $x = x_1 v_1 + \dots + x_n v_n$ . For general bases  $|\text{component}| \neq \|x\|$ , but,  
if we can prove in nice basis, [P24] helps us transfer our result to other coord. system....

P25  $F, G: V \rightarrow \mathbb{R}$  and  $\beta = \{v_1, \dots, v_n\}$  a basis for  $V$  with coordinates  $x_1, \dots, x_n$ . Also,  $h: \mathbb{R} \rightarrow \mathbb{R}$  a diff. function,  $c \in \mathbb{R}$  and  $i = 1, 2, \dots, n$ , Using linearity of derivative on  $\mathbb{R}$ ,

$$\begin{aligned}\frac{\partial}{\partial x_i} [cF(x) + G(x)] &= \left. \frac{d}{dt} [cF(x+tv_i) + G(x+tv_i)] \right|_{t=0} \\ &= c \left. \frac{d}{dt} [F(x+tv_i)] \right|_{t=0} + \left. \frac{d}{dt} [G(x+tv_i)] \right|_{t=0} \\ &= c \underbrace{\frac{\partial F}{\partial x_i}}_{\text{---}} + \underbrace{\frac{\partial G}{\partial x_i}}_{\text{---}}.\end{aligned}$$

Likewise, using chain-rule for functions on  $\mathbb{R}$ ,

$$\begin{aligned}\frac{\partial}{\partial x_i} [h(F(x))] &= \left. \frac{d}{dt} [h(F(x+tv_i))] \right|_{t=0} \\ &= \left( h'(F(x+tv_i)) \left. \frac{d}{dt} [F(x+tv_i)] \right|_{t=0} \right) \\ &= h'(F(x)) \left. \frac{d}{dt} [F(x+tv_i)] \right|_{t=0} \\ &= h'(F(x)) \underbrace{\frac{\partial F}{\partial x_i}}_{\text{---}}.\end{aligned}$$

P26 Let  $F: \mathbb{R}^{n \times n} \xrightarrow{\quad} \mathbb{R}^{n \times n}$  be defined by  $F(A) = A^3$

$$\begin{aligned}(a.) F(A+H) - F(A) &= (A+H)^3 - A^3 \\ &= (A+H)(A^2 + HA + AH + H^2) - A^3 \\ &= A^3 + \underbrace{AHA + A^2H + HA^2 + AH^2 + H^2A + HAH + H^3 - A^3}_{dF_A(H)} \leftarrow \begin{matrix} \text{I leave} \\ \text{show this linear next page} \end{matrix} \Rightarrow\end{aligned}$$

Consider,

$$\frac{F(A+H) - F(A) - dF_A(H)}{\|H\|} = \frac{1}{\|H\|} (AH^2 + H^2A + HAH + H^3) = \eta$$

Notice, for  $H \neq 0$ ,  $0 \leq \|\eta\| \leq \frac{1}{\|H\|} (\|A\| \|H\|^2 + \|H\|^2 \|A\| + \|H\| \|A\| \|H\| + \|H\|^3)$

thus  $0 \leq \|\eta\| \leq 3\|A\| \|H\| + \|H\|^2 \rightarrow 0$  as  $H \rightarrow 0$

implies  $\|H\| \rightarrow 0$  thus, by Squeeze Thm,  $\lim_{H \rightarrow 0} \|\eta\| = 0$  and

therefore  $\lim_{H \rightarrow 0} \eta = 0$ , and it follows  $F$  is diff at  $A$  with  $dF_A(H)$  as claimed.  $\square$

One last item, we should check linearity of  $dF_A$ ,

$$\begin{aligned} dF_A(H_1 + H_2) &= A(cH_1 + H_2)A + A^2(cH_1 + H_2) + (cH_1 + H_2)A^2 \\ &= c(AH_1 A + A^2 H_1 + H_1 A^2) + AH_2 A + A^2 H_2 + H_2 A^2 \\ &= c dF_A(H_1) + dF_A(H_2). \end{aligned}$$

Thus  $dF_A$  is linear as claimed. Thus,  $d_A F_A(H) = AHA + A^2H + HA^2$

$$(6.) dF_A(H) = dF_A\left(\sum_{i,j} H_{ij} E_{ij}\right) = \sum_{i,j} H_{ij} \underbrace{dF_A(E_{ij})}_{\frac{\partial F}{\partial X_{ij}}} \quad (\text{or you might say } \frac{\partial F}{\partial A_{ij}})$$

Following  $\approx$  minutes 10-12 of part 2 of Lecture 3 of Fall 2015,

$$\begin{aligned} \frac{\partial F}{\partial X_{ij}}(A) &= \frac{d}{dt} \left[ F(A + tE_{ij}) \right] \Big|_{t=0} \\ &= \frac{d}{dt} \left[ (A + tE_{ij})^3 \right] \Big|_{t=0} \\ &= \frac{d}{dt} \left[ (A + tE_{ij})(A^2 + t(E_{ij}A + AE_{ij}) + t^2E_{ij}^2) \right] \Big|_{t=0} \\ &= \frac{d}{dt} \left[ A^3 + t(AE_{ij}A + A^2E_{ij} + E_{ij}A^2) + \underbrace{t^2(G) + t^3(A)}_{\text{vanish after next step as residual } t \text{ evaluates to zero.}} \right] \Big|_{t=0} \\ &= \underline{AE_{ij}A + A^2E_{ij} + E_{ij}A^2} \end{aligned}$$

Then, by Thm on continuously diff  
 $\Rightarrow$  Frechet diff. we derive,

$$\begin{aligned} d_A F(H) &= \sum_{i,j} H_{ij} (AE_{ij}A + A^2E_{ij} + E_{ij}A^2) \\ &= A \left( \sum_{i,j} H_{ij} E_{ij} \right) A + A^2 \left( \sum_{i,j} H_{ij} E_{ij} \right) + \left( \sum_{i,j} H_{ij} E_{ij} \right) A^2 \\ &= \underline{AHA + A^2H + HA^2} \end{aligned}$$

Remark:  $\frac{\partial}{\partial X_{ij}}(A^3) \neq 3A^2 \frac{\partial A}{\partial X_{ij}}$  we have to face the departure from commutativity!

P27 If  $F, G : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  are diff at  $A$  then  
 $FG$  is likewise diff. at  $A$

If  $F$  and  $G$  are diff at  $A$  then  $R_F(H), R_G(H) \rightarrow 0$   
as  $H \rightarrow 0$  where we define, for  $H \neq 0$ ,

$$R_F(H) = \frac{F(A+H) - F(A) - d_A F(H)}{\|H\|}$$

$$R_G(H) = \frac{G(A+H) - G(A) - d_A G(H)}{\|H\|}$$

Thus,  $F(A+H) = F(A) + d_A F(H) + \|H\| R_F(H)$

and  $G(A+H) = G(A) + d_A G(H) + \|H\| R_G(H)$ . Consider,

$$\begin{aligned} F(A+H)G(A+H) &= (F(A) + d_A F(H) + \|H\| R_F(H))(G(A) + d_A G(H) + \|H\| R_G(H)) \\ &\stackrel{\downarrow}{=} F(A)G(A) + d_A F(H)G(A) + F(A)d_A G(H) + \|H\| [R_F d_A G(H) + \\ &\quad \curvearrowright + R_F R_G \|H\| +] \\ &\quad \curvearrowright + d_A F(H)R_G(H) +] \\ &\quad \curvearrowright + d_A F(H)d_A G(H) +] \\ &\quad \curvearrowright + F(A)\|H\| R_G(H) +] \\ &\quad \curvearrowright + \|H\| R_F(H)G(A) \end{aligned}$$

Sorry, a little messy, but I  
hope you can follow,

$$\begin{aligned} R_{FG}(H) &= \frac{F(A+H)G(A+H) - F(A)G(A) - d_A F(H)G(A) - F(A)d_A G(H)}{\|H\|} \\ &= R_F(H)d_A G(H) + R_F(H)R_G(H) + \underbrace{d_A F\left(\frac{H}{\|H\|}\right)R_G(H)}_{*} + d_A F\left(\frac{H}{\|H\|}\right)d_A G(H) + \\ &\quad \curvearrowright + F(A)R_G(H) + R_F(H)G(A) \end{aligned}$$

Hence  $R_{FG}(H) \rightarrow 0$  as  $H \rightarrow 0$  since each of the  
terms above vanishes as  $R_F(H), R_G(H) \rightarrow 0$ . To understand  $*$   
notice  $\left\|\frac{H}{\|H\|}\right\| = \frac{\|H\|}{\|H\|} = 1$  thus  $\left\|d_A F\left(\frac{H}{\|H\|}\right)\right\| \leq M$  since each  
component func of  $d_A F$  attains max on compact domain then

$$\|*\| = \left\|d_A F\left(\frac{H}{\|H\|}\right)[R_G(H) + d_A G(H)]\right\| \leq M \|R_G(H) + d_A G(H)\| \rightarrow 0$$

thus  $* \rightarrow 0$  by Squeeze Thm. Finally, observe

$$(d_A FG)(H) = d_A F(H)G(A) + F(A)d_A G(H) \text{ is linear } \therefore FG \text{ is diff. at } A.$$

P28 Let  $f(x, y) = \sqrt[y]{x}$  and  $\gamma(t) = (t, t)$ . Calc.  $\frac{d}{dt} [\sqrt[t]{t}]$ .

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^{\frac{1}{y}}) = \frac{1}{y} x^{\frac{1}{y}-1} = \frac{1}{y} x^{\frac{1-y}{y}} = \frac{1}{y} x^{\frac{1}{y}x^{-1}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^{\frac{1}{y}}) = \ln(x) x^{\frac{1}{y}} \frac{\partial}{\partial y} \left[ \frac{1}{y} \right] = \ln(x) \frac{x^{\frac{1}{y}}}{-y^2} = \frac{\ln(x) x^{\frac{1}{y}}}{-y^2}.$$

Thus,

$$\begin{aligned} \frac{d}{dt} [\sqrt[t]{t}] &= \frac{d}{dt} [f(\gamma(t))] = \nabla f(\gamma(t)) \cdot \gamma'(t) \\ &= \langle f_x(t, t), f_y(t, t) \rangle \cdot \langle 1, 1 \rangle \\ &= \left\langle \frac{1}{t} t^{\frac{1-t}{t}}, \frac{\ln(t)t^{\frac{1}{t}}}{-t^2} \right\rangle \cdot \langle 1, 1 \rangle \\ &= \left\langle \frac{1}{t^2} t^{\frac{1}{t}}, \frac{-1}{t^2} \ln(t) t^{\frac{1}{t}} \right\rangle \cdot \langle 1, 1 \rangle \\ &= \boxed{\frac{1}{t^2} t^{\frac{1}{t}} (1 - \ln(t))} = \frac{\sqrt[t]{t}}{t^2} (1 - \ln(t)). \end{aligned}$$

P29 Suppose  $\bar{x}(\bar{s}, \bar{t}) = x(s, t)$  then chain rule yields

$$\frac{\partial \bar{x}}{\partial s} = \frac{\partial}{\partial s} (\bar{x}(\bar{s}, \bar{t})) = \underbrace{\frac{\partial \bar{x}}{\partial \bar{s}} \frac{\partial \bar{s}}{\partial s} + \frac{\partial \bar{x}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial s}}.$$

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial}{\partial t} (\bar{x}(\bar{s}, \bar{t})) = \underbrace{\frac{\partial \bar{x}}{\partial \bar{s}} \frac{\partial \bar{s}}{\partial t} + \frac{\partial \bar{x}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t}}.$$

[PROBLEM 29] (again !)

$$\frac{\partial \bar{X}}{\partial s} = \frac{\partial}{\partial s} (\bar{X}(s, t), \bar{t}(s, t))$$

$$\frac{\partial \bar{X}}{\partial s} = \frac{\partial \bar{X}}{\partial \bar{s}} \frac{\partial \bar{s}}{\partial s} + \frac{\partial \bar{X}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial s}$$

Likewise

$$\frac{\partial \bar{X}}{\partial t} = \frac{\partial \bar{X}}{\partial \bar{s}} \frac{\partial \bar{s}}{\partial t} + \frac{\partial \bar{X}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t}$$

[PROBLEM 30]

$$\begin{aligned} \frac{\partial \bar{X}}{\partial s} \times \frac{\partial \bar{X}}{\partial t} &= \left( \frac{\partial \bar{X}}{\partial \bar{s}} \frac{\partial \bar{s}}{\partial s} + \frac{\partial \bar{X}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial s} \right) \times \left( \frac{\partial \bar{X}}{\partial \bar{s}} \frac{\partial \bar{s}}{\partial t} + \frac{\partial \bar{X}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t} \right) \\ &= \cancel{\left( \frac{\partial \bar{X}}{\partial \bar{s}} \times \frac{\partial \bar{X}}{\partial \bar{s}} \right)}^0 \frac{\partial \bar{s}}{\partial s} \frac{\partial \bar{s}}{\partial t} + \cancel{\left( \frac{\partial \bar{X}}{\partial \bar{s}} \times \frac{\partial \bar{X}}{\partial \bar{t}} \right)}^0 \frac{\partial \bar{s}}{\partial s} \frac{\partial \bar{t}}{\partial t} + \cancel{2} \\ &\quad + \cancel{\left( \frac{\partial \bar{X}}{\partial \bar{t}} \times \frac{\partial \bar{s}}{\partial \bar{s}} \right)}^0 \frac{\partial \bar{t}}{\partial s} \frac{\partial \bar{s}}{\partial t} + \cancel{\left( \frac{\partial \bar{X}}{\partial \bar{t}} \times \frac{\partial \bar{s}}{\partial \bar{t}} \right)}^0 \frac{\partial \bar{t}}{\partial s} \frac{\partial \bar{t}}{\partial t} \\ &= \left( \frac{\partial \bar{X}}{\partial \bar{s}} \times \frac{\partial \bar{X}}{\partial \bar{t}} \right) \left( \frac{\partial \bar{s}}{\partial s} \frac{\partial \bar{t}}{\partial t} - \frac{\partial \bar{t}}{\partial s} \frac{\partial \bar{s}}{\partial t} \right) \end{aligned}$$

Consider,

$$\iint_D \vec{F}(\bar{X}(s, t)) \cdot \left( \frac{\partial \bar{X}}{\partial s} \times \frac{\partial \bar{X}}{\partial t} \right) ds dt = \cancel{2}$$

$$\iint_D \vec{F}(\bar{X}(s, t)) \cdot \left( \frac{\partial \bar{X}}{\partial \bar{s}} \times \frac{\partial \bar{X}}{\partial \bar{t}} \right) \underbrace{\left( \frac{\partial \bar{s}}{\partial s} \frac{\partial \bar{t}}{\partial t} - \frac{\partial \bar{t}}{\partial s} \frac{\partial \bar{s}}{\partial t} \right)}_{\text{need this is positive}} ds dt$$

$$\iint_{\bar{D}} \vec{F}(\bar{X}(s, t)) \cdot \left( \frac{\partial \bar{X}}{\partial \bar{s}} \times \frac{\partial \bar{X}}{\partial \bar{t}} \right) d\bar{s} d\bar{t}$$

so we can replace it  
with  $\left| \frac{\partial \bar{s}}{\partial s} \frac{\partial \bar{t}}{\partial t} - \frac{\partial \bar{t}}{\partial s} \frac{\partial \bar{s}}{\partial t} \right|$

which appears in  
the multivariate  
change of variables  
for integrals theorem.

To obtain the equality here  
we need  $\det(T') > 0$  as

$$T' = \begin{bmatrix} \bar{s}/s & \bar{t}/s \\ \bar{s}/t & \bar{t}/t \end{bmatrix}$$

this makes  $\bar{X}$  and  $\bar{t}$  share the same  
orientation. Or, they're compatible patches for oriented  $S$ .

**PROBLEM 31** Suppose  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is invertible on  $U \subseteq \mathbb{R}^n$  then  $F \circ F^{-1} = \text{id}_{F(U)}$  and  $F^{-1} \circ F = \text{id}_U$ .

By the chain-rule and the fact.  $d(\text{id}) = \text{id}$  as  $\text{id}$  is linear transformation we find

$$dF \circ dF^{-1} = \text{id} \quad \text{and} \quad dF^{-1} \circ dF = \text{id}$$

Hence  $dF$  is invertible at each point.

$$\text{Btw, } d(F^{-1} \circ F)_{x_0} = (dF^{-1})_{F(x_0)} \circ dF_{x_0} = \text{Id}$$

I'm not sure including the point-dependence adds much here...

**PROBLEM :** Edwards #3.6 on page 88

$$\text{Suppose } \bar{\Phi} = 5 \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 u}{\partial y^2} \Rightarrow \bar{\Phi} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}$$

given  $x = 2s+t$ ,  $y = s-t$ , Begin by stating what this means in terms of functions.

$$\bar{\Phi} = \bar{\Phi}(x(s,t), y(s,t))$$

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial s} [u(x(s,t), y(s,t))] = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} = 2u_x + u_y$$

$$\frac{\partial^2 u}{\partial s^2} = \left[ 2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right] (2u_x + u_y) = 4u_{xx} + 2u_{yx} + 2u_{xy} + u_{yy}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} = u_x - u_y$$

$$\frac{\partial^2 u}{\partial t^2} = \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right] [u_x - u_y] = u_{xx} - u_{yx} - u_{xy} + u_{yy}$$

$$\text{Thus, } \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = 5 \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 u}{\partial y^2}$$

Remark: This sol<sup>n</sup> added for fun... oh maybe laziness.

PROBLEM 32 # 3.11 pg 89 Edwards

(a.) If  $f(\vec{x}) = g(r)$  where  $r = \|\vec{x}\|$  and  $n \geq 3$  show

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_n^2} = \frac{n-1}{r} g'(r) + g''(r) \text{ for } \vec{x} \neq 0.$$

(b.) If  $\nabla^2 f = 0 \Rightarrow f(\vec{x}) = \frac{a}{r^{n-2}} + b$  for  $\vec{x} \neq 0$ ,  $a, b \in \mathbb{R}$

(a.) Observe  $\frac{\partial r}{\partial x_j}$  is easily calculated from  $\vec{x} \cdot \vec{x} = r^2$

$$\vec{x} \cdot \vec{x} = \sum_{i=1}^n x_i^2 = r^2. \text{ Note, } \frac{\partial}{\partial x_j} \left( \sum_{i=1}^n x_i^2 \right) = \sum_{i=1}^n 2x_i \delta_{ij} = 2x_j.$$

$$\text{thus } 2r \frac{\partial r}{\partial x_j} = 2x_j \Rightarrow \frac{\partial r}{\partial x_j} = \frac{x_j}{r} \text{ for } j=1, 2, \dots, n.$$

$$\text{Further, } \frac{\partial g}{\partial x_j} = g'(r) \frac{\partial r}{\partial x_j} = \frac{g'(r)}{r} x_j$$

$$\begin{aligned} \text{Thus } \frac{\partial^2 g}{\partial x_j^2} &= \frac{\partial}{\partial x_j} \left[ g'(r) \frac{x_j}{r} \right] = g''(r) \frac{\partial r}{\partial x_j} \frac{x_j}{r} + g'(r) \\ &= g''(r) \frac{\partial r}{\partial x_j} \frac{x_j}{r} + g'(r) \frac{\partial}{\partial x_j} \left( \frac{x_j}{r} \right) \\ &= g''(r) \frac{x_j^2}{r^2} + g'(r) \left[ \frac{r - x_j \left( \frac{x_j}{r} \right)}{r^2} \right] \\ &= g''(r) \frac{x_j^2}{r^2} + g'(r) \left[ \frac{r^2 - x_j^2}{r^3} \right] \end{aligned}$$

Therefore,

$$\begin{aligned} \nabla^2 f &= g''(r) \left[ \frac{x_1^2 + \dots + x_n^2}{r^2} \right] + \frac{g'(r)}{r^3} \left[ nr^2 - x_1^2 - \dots - x_n^2 \right] \\ &= g''(r) + \frac{g'(r)}{r^3} [nr^2 - r^2] \\ &= g''(r) + \underline{\frac{n-1}{r} g'(r)} \end{aligned}$$

$$(b.) g''(r) = \frac{1-n}{r} g'(r) \rightarrow \frac{dv}{v} = \frac{(1-n)dr}{r} \Rightarrow \ln|v| = (1-n) \ln|r| + C,$$

$$\frac{dv}{dr} = \frac{1-n}{r} v \quad (v = g'(r))$$

reduction of order  
technique

$$\Rightarrow \frac{dg}{dr} = \tilde{a} r^{1-n}$$

$$\Rightarrow g(r) = \frac{\tilde{a} r^{2-n}}{2-n} + b$$

$$\therefore f(r) = \frac{a}{r^{n-2}} + b$$

P33 problem 1.36 on p. 23 of Renteln

$$\det(A) = \sum_{i_1, \dots, i_n=1}^n \epsilon_{i_1 i_2 \dots i_n} A_{i_1 1} A_{i_2 2} \dots A_{i_n n}$$

$$\begin{aligned} \frac{\partial}{\partial A_{ij}} [\det(A)] &= \sum_{i_1, \dots, i_n} \epsilon_{i_1 \dots i_n} \left( \underbrace{\frac{\partial A_{i_1 1}}{\partial A_{ij}} A_{i_2 2} \dots A_{i_n n}}_{\delta_{i_1, i} \delta_{i_2, j}} + \underbrace{A_{i_1 1} \frac{\partial A_{i_2 2}}{\partial A_{ij}} \dots A_{i_n n} + \dots + A_{i_1 1} \dots A_{i_{n-1} n-1} \frac{\partial A_{i_n n}}{\partial A_{ij}}} \right) \\ &= \sum \epsilon_{i_1 \dots i_n} \delta_{i_1, i} A_{i_2 2} \dots A_{i_n n} + \dots \\ &\quad + \sum \epsilon_{i_1 \dots i_n} \delta_{i_2, j} A_{i_1 1} A_{i_3 3} \dots A_{i_n n} + \dots \\ &\quad + \sum \epsilon_{i_1 \dots i_n} \delta_{i_n, n} A_{i_1 1} \dots A_{i_{n-1} n-1} \end{aligned}$$

If we examine the expression above then after some thought perhaps you can convince yourself it is  $\tilde{A}_{ij} = (-1)^{i+j} \det(A(i/j))$ . However, it's way easier to take my own advice, for any  $i$ ,

$$\det(A) = \sum_{j=1}^n A_{ij} \tilde{A}_{ij} \quad (\epsilon_i = 1.57)$$

Notice  $\tilde{A}_{ij}$  is not a function of  $A_{ij}$  !

Thus  $\frac{\partial \tilde{A}_{ij}}{\partial A_{ij}} = 0$  and so, for any  $l$ ,

$$\begin{aligned} \frac{\partial}{\partial A_{ij}} (\det(A)) &= \frac{\partial}{\partial A_{ij}} \left( \sum_{k=1}^n A_{ik} \tilde{A}_{kh} \right) \\ &= \sum_{k=1}^n \left( \underbrace{\frac{\partial A_{ik}}{\partial A_{ij}} \tilde{A}_{kh}}_{\delta_{ki} \delta_{kj}} + A_{ik} \frac{\partial \tilde{A}_{kh}}{\partial A_{ij}} \right) \\ &= \tilde{A}_{lj} \delta_{li} + \sum_k A_{ik} \frac{\partial \tilde{A}_{kh}}{\partial A_{ij}} \end{aligned}$$

\*) why? see ↗

Take  $l=i$  to see  $\frac{\partial}{\partial A_{ij}} [\det(A)] = \tilde{A}_{ii}$  as  $\frac{\partial \tilde{A}_{ik}}{\partial A_{ij}} = 0$ .

P 33 continued

- $\tilde{A}_{ik} = \det(A(i/k))(-1)^{i+k}$  ← determinant of matrix A with row i and column k deleted  
Has no  $A_{il}, \dots, A_{in}$  and no  $A_{ik}, A_{jk}, \dots, A_{nk}$  dependence.

$$\therefore \frac{\partial \tilde{A}_{ik}}{\partial A_{il}} = 0 \quad \text{and} \quad \frac{\partial \tilde{A}_{ik}}{\partial A_{lk}} = 0 \quad \text{for any } l=1, 2, \dots, n$$

which is what we needed to finish the calculation at \*.

$$\therefore \boxed{\frac{\partial}{\partial A_{ij}} [\det(A)] = \tilde{A}_{ij} = (-1)^{i+j} \det(A(i|j))}$$

As shown in 1.33 or linear algebra,

$$A \text{adj}(A) = \det(A) I \quad \text{where } \text{adj}(A)_{ij} = \tilde{A}_{ji}$$

$$\Rightarrow \text{adj}(A) = \det(A) A^{-1}$$

$$\Rightarrow \tilde{A}_{ij} = \det(A) (A^{-1})_{ji}$$

$$\therefore \boxed{\frac{\partial}{\partial A_{ij}} [\det(A)] = \det(A) (A^{-1})_{ji}}$$

[P34] Problem 1.46 of Renth pg. 25

$$e_k(x_1, \dots, x_n) \stackrel{\text{def}}{=} \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k} \quad \text{for } k \geq 1$$

and we set  $e_0 = 1$ . For example, in  $n=3$  case,

$$e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

$$e_3(x_1, x_2, x_3) = x_1 x_2 x_3$$

} elementary symmetric functions

We also define the power sum symmetric function

$$P_k(x_1, \dots, x_n) \stackrel{\text{def}}{=} \sum_{i=1}^n x_i^k \quad \begin{array}{l} P_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 \\ P_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \\ P_3(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 \end{array}$$

Notationally setting these in  
 $\infty$ -dim'l set-up

$$e_k \stackrel{\text{def}}{=} \sum_{1 \leq i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad P_k \stackrel{\text{def}}{=} \sum_{i \geq 1} x_i^k$$

then to obtain sym. frnts in  $n$ -dimensions just set  
 $x_{n+1}, x_{n+2}, \dots = 0$ .

(a.) Let  $E(t) = \sum_{j=1}^{\infty} e_j t^j$ . Show  $E(t) = \prod_{j=1}^{\infty} (1 + x_j t)$

Consider,

$$1 + x_1 t$$

$$(1 + x_1 t)(1 + x_2 t) = 1 + (x_1 + x_2)t + x_1 x_2 t^2$$

$$(1 + x_1 t)(1 + x_2 t)(1 + x_3 t) = 1 + (x_1 + x_2 + x_3)t + (x_1 x_2 + x_1 x_3 + x_2 x_3)t^2 + x_1 x_2 x_3 t^3$$

⋮

$$E(t) = e_0 + e_1 t + \dots + e_n t^n = \prod_{j=0}^n (1 + x_j t)$$

Remark: perhaps I can improve this with  
 an inductive argument,

$$\text{Assume } E_n(t) = e_0 + e_1 t + \cdots + e_n t^n = \prod_{j=1}^n (1 + x_j t)$$

$$\text{Consider, } E_{n+1}(t) = \tilde{e}_0 + \tilde{e}_1 t + \cdots + \tilde{e}_n t^n + \tilde{e}_{n+1} t^{n+1}, \quad \begin{array}{l} (\tilde{e}_j \text{ in } (n+1)\text{-variables.}) \\ (e_j \text{ in } n\text{-variables}) \end{array}$$

$$\begin{aligned} \prod_{j=1}^{n+1} (1 + x_j t) &= \left( \prod_{j=1}^n (1 + x_j t) \right) (1 + x_{n+1} t) \\ &= (e_0 + e_1 t + \cdots + e_n t^n)(1 + x_{n+1} t) \\ &= e_0 + (e_1 + x_{n+1})t + (e_2 + e_1 x_{n+1})t^2 + \cdots + e_n x_{n+1} t^{n+1} \\ &= \tilde{e}_0 + \tilde{e}_1 t + \cdots + \tilde{e}_{n+1} t^{n+1} \\ &= E_{n+1}(t). \Rightarrow \text{claim true } \forall n \in \mathbb{N}. \text{ We already established } n=1, 2 \text{ and } 3 \text{ last page.} \end{aligned}$$

$$(6.) P(t) \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} p_j t^{j-1}, \text{ show } P(t) = \sum_{j=1}^{\infty} \frac{x_j}{1 - x_j t}$$

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{x_j}{1 - x_j t} &= \sum_{j=1}^{\infty} x_j (1 + (x_j t) + (x_j t)^2 + (x_j t)^3 + \cdots) \\ &= \sum_{j=1}^{\infty} (x_j + x_j^2 t + x_j^3 t^2 + x_j^4 t^3 + \cdots) = p_1 + p_2 t + \cdots \end{aligned}$$

Compare to,

$$P(t) = \sum_{j=1}^{\infty} p_j t^{j-1} = p_1 + p_2 t + p_3 t^2 + \cdots \quad \text{yep.}$$

$$(c.) \text{ Observe, } \ln E = \ln \prod_{j=1}^{\infty} (1 + x_j t) = \sum_{j=1}^{\infty} \ln (1 + x_j t)$$

$$\Rightarrow \frac{1}{E} \frac{dE}{dt} = \sum_{j=1}^{\infty} \frac{x_j}{1 + x_j t} = P(-t)$$

$$\Rightarrow EP(-t) = \frac{dE}{dt} = \sum_{j=1}^{\infty} j t^{j-1} e_j$$

$$(e_0 + e_1 t + \cdots)(p_1 - p_2 t + p_3 t^2 + \cdots) = e_1 + 2e_2 t + 3e_3 t^2 + \cdots$$

$$\text{Equating coeff. of } t^{k-1} \text{ yields } k e_k = \sum_{i=1}^{k-1} (-1)^{i-1} e_{k-i} p_i$$

$$\text{For example, } e_1 = e_0 p_1, 2e_2 = e_1 p_1 - e_0 p_2 \text{ etc...}$$

Remark: for  $\mathbb{R}^n$   
these are all  
finite sums & products.  
No convergence  
issue here!

$$(d.) k e_k = \sum_{i=1}^k (-1)^{i-1} e_{k-i} p_i$$

I think he means to write identity for  $k=1, 2, \dots, n$

$$e_1 = e_0 p_1 = p_1$$

$$2e_2 = -e_0 p_2 + e_1 p_1 \rightarrow p_1 e_1 - 2e_2 = p_2$$

$$3e_3 = e_0 p_3 - e_1 p_2 + e_2 p_1 \rightarrow -p_2 e_1 + p_1 e_2 - 3e_3 = -p_3$$

$$4e_4 = -e_0 p_4 + e_1 p_3 - e_2 p_2 + e_3 p_1 \\ \vdots$$

$$n e_n = e_0 p_n (-1)^{n-1} - e_1 p_{n-1} (-1)^{n-2} + \dots + e_{n-1} p_1$$

$$\hookrightarrow (-1)^{n-1} p_{n-1} e_1 + \dots - e_{n-1} p_1 + n e_n = (-1)^{n-1} p_n$$

Hence,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ p_1 & -2 & \cdots & 0 \\ -p_2 & p_1 & -3 & \cdots & 0 \\ \vdots & & & & \\ (-1)^{n-1} p_{n-1} & \cdots & -p_1 & n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ -p_3 \\ \vdots \\ (-1)^{n-1} p_n \end{bmatrix}$$

$M = [\bar{M} | M_n]$        $b$

Cramer's Rule:

$$e_n = \frac{\det [\bar{M} | b]}{\det (M)} = \frac{\det (\bar{M} | b)}{n!}$$

$$\det \begin{bmatrix} 1 & 0 & \cdots & p_1 \\ p_1 & -2 & \cdots & p_2 \\ -p_2 & p_1 & -3 & \cdots & -p_3 \\ \vdots & & & & \\ (-1)^{n-1} p_{n-1} & \cdots & -p_1 & p_n (-1)^{n-1} \end{bmatrix} = \det \begin{bmatrix} p_1 & 1 & 0 & \cdots \\ p_2 & p_1 & -2 & \cdots \\ -p_3 & \vdots & p_1 & \ddots \\ \vdots & & & \vdots \\ p_n (-1)^{n+1} & p_{n-1} & \vdots & \vdots \end{bmatrix}$$

now, a few row operations  
gets us to the desired  
formula, sorry, but, I stop here.

(e.) Notice  $\det A = \lambda_1 \lambda_2 \cdots \lambda_n = e_n(\lambda_1, \dots, \lambda_n)$

and  $\text{trace}(A) = \text{tr}(A) = \lambda_1 + \cdots + \lambda_n = e_1(\lambda_1, \dots, \lambda_n) = P_1(\lambda_1, \dots, \lambda_n)$

However, as  $AV = \lambda V \Rightarrow A^k V = \lambda^k V$  it follows that

$A^k$  has e-values  $\lambda_1^k, \dots, \lambda_n^k \Rightarrow \text{tr}(A^k) = \lambda_1^k + \cdots + \lambda_n^k = P_k(\lambda_1, \dots, \lambda_n)$ .

We can use the result of (d.),

$$\begin{aligned} e_4 &= \frac{1}{4!} \det \begin{bmatrix} P_1 & 1 & 0 & 0 \\ P_2 & P_1 & 2 & 0 \\ P_3 & P_2 & P_1 & 3 \\ P_4 & P_3 & P_2 & P_1 \end{bmatrix} \\ &= \frac{1}{4!} \left( P_1 \det \begin{bmatrix} P_2 & 2 & 0 \\ P_3 & P_1 & 3 \\ P_4 & P_2 & P_1 \end{bmatrix} - \det \begin{bmatrix} P_2 & 2 & 0 \\ P_3 & P_1 & 3 \\ P_4 & P_2 & P_1 \end{bmatrix} \right) \\ &= \frac{1}{4!} \left( P_1 \left[ P_1(P_1^2 - 3P_2) - 2(P_1P_2 - 3P_3) \right] - P_2(P_1^2 - 3P_2) + 2(P_1P_3 - 3P_4) \right) \\ &= \frac{1}{4!} \left( P_1^4 - \underline{3P_1^2P_2} - \underline{2P_1^2P_2} + \underline{6P_1P_3} - \underline{P_1^2P_2} + \underline{3P_2^2} + \underline{2P_1P_3} - \underline{6P_4} \right) \\ &= \frac{1}{4!} (P_1^4 - 6P_1^2P_2 + 8P_1P_3 + 3P_2^2 - 6P_4) \end{aligned}$$

$$\therefore \boxed{\det(A) = \frac{1}{4!} \left( (\text{tr}(A))^4 - 6(\text{tr}(A))^2 \text{tr}(A^2) + 8\text{tr}(A)\text{tr}(A^3) + 3(\text{tr}(A^2))^2 - 6\text{tr}(A^4) \right)}$$

Also,

$$e_3 = \frac{1}{3!} \det \begin{bmatrix} P_1 & 1 & 0 \\ P_2 & P_1 & 2 \\ P_3 & P_2 & P_1 \end{bmatrix} = \frac{1}{3!} (P_1^3 - 2P_1P_2 - P_1P_2 + 2P_3)$$

$$\boxed{\det(A) = \frac{1}{3!} \left( \text{tr}(A)^3 - 3\text{tr}(A)\text{tr}(A^2) + 2\text{tr}(A^3) \right)}$$

- Both results above stem from viewing det. & trace as symmetric functions of the e-values. See the top of page for more.

[P35] P1.49, The Cauchy Schwarz Inequality

Let  $g: V \times V \rightarrow \mathbb{R}$  be an inner-product over a real vector space  $V$ . Let  $x, y$  be vectors in  $V$  and consider  $f(t) = g(x+ty, x+ty)$ .

Observe  $g(x+ty, x+ty) = 0 \iff x+ty = 0 \iff x = -ty$ . \*

Generally  $f(t) \geq 0$  and we can calculate,

$$f(t) = g(x, x) + 2t g(x, y) + t^2 g(y, y)$$

$$f(t) = C + Bt + At^2 \text{ with } A = g(y, y), B = 2g(x, y), C = g(x, x).$$

Notice  $C + Bt + At^2 \geq 0 \Rightarrow$  either  $C + Bt + At^2 = 0$  (already covered at \*) or  $C + Bt + At^2 > 0$  for all  $t \in \mathbb{R}$

and  $\underbrace{B^2 - 4AC < 0}$  (think about geometry of quadratic function graph)

$$4g(x, y)g(x, y) - 4g(x, x)g(y, y) < 0$$

$$\Rightarrow \boxed{(g(x, y))^2 < g(x, x)g(y, y) \text{ for } x \neq -ty}$$

$$\Rightarrow |g(x, y)| < \sqrt{g(x, x)} \sqrt{g(y, y)} \hookrightarrow |g(x, y)| < \underbrace{\|x\| \|y\|}_{\text{if } \|x\| = \sqrt{g(x, x)}}$$

At \* we saw  $g(x+ty, x+ty) = 0 \text{ if } x = -ty$  if  $\|x\| = \sqrt{g(x, x)}$

$$\begin{aligned} \text{Notice, } x = -ty &\Rightarrow (g(x, y))^2 = (g(-ty, y))^2 \\ &= t^2 g(y, y) g(y, y) \\ &= g(-ty, -ty) g(y, y) \\ &= g(x, x) g(y, y). \end{aligned}$$

PROBLEM 36

(a.) Let  $\langle , \rangle : V \times V \rightarrow \mathbb{R}$  be an inner-product and  $\|x\| = \sqrt{\langle x, x \rangle}$ . Consider,

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \quad ①$$

$$\|x-y\|^2 = \langle x-y, x-y \rangle = \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle \quad ②$$

Therefore, as  $\langle x, x \rangle = \|x\|^2$  and  $\langle y, y \rangle = \|y\|^2$  adding the above yields,

$$\boxed{\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 = 2(\|x\|^2 + \|y\|^2)} \quad (*).$$

Remark: There was a typo in Curtis' Abstract Linear Algebra from which I crafted this problem.

(b.) Likewise, if we take the difference of ① and ②,

$$\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y \rangle$$

$$\Rightarrow \boxed{\langle x, y \rangle = \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2)}$$

(c.) Define  $g(x, y) = \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2)$  for  $\|\cdot\|$  a norm on  $V$ ,

$$\begin{aligned} (i.) \quad g(x, y) &= \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2) \\ &= \frac{1}{4} (\|y+x\|^2 - \|y-x\|^2) = \text{ as } \|x-y\| = \|y-x\|. \\ &= g(y, x). \end{aligned}$$

$$\begin{aligned} (ii.) \quad g(x, x) &= \frac{1}{4} (\|x+x\|^2 - \|x-x\|^2) \\ &= \frac{1}{4} \|2x\|^2 \\ &= \|x\|^2 \geq 0 \quad \text{and} \quad g(x, x) = \|x\|^2 = 0 \text{ iff } x = 0. \end{aligned}$$

Remark: I tried to prove (iii.) w/o using (\*).

Trust me, it's really, really hard. 😊

P36 continued

(iii) We wish to show  $g(x, z) + g(y, z) = g(x+y, z)$ .

Assume  $\underbrace{\|A+B\|^2 + \|A-B\|^2}_{\|A+B\|^2 = \underbrace{2\|A\|^2 + 2\|B\|^2 - \|A-B\|^2} = 2\|A\|^2 + 2\|B\|^2}$ . Consider

$$4g(x, z) = \|x+z\|^2 - \|x-z\|^2 = 2\|x\|^2 + 2\|z\|^2 - 2\|x-z\|^2$$

$$4g(y, z) = \|y+z\|^2 - \|y-z\|^2 = 2\|y\|^2 + 2\|z\|^2 - 2\|y-z\|^2$$

Otherwise,

$$\begin{aligned} 4g(x+y, z) &= \|x+y+z\|^2 - \|x+y-z\|^2 \\ &= \|(x+z)+y\|^2 - \|x+(y-z)\|^2 \\ &= 2\|x+z\|^2 + 2\|y\|^2 - \cancel{\|x+z-z\|^2} - 2\|x\|^2 - 2\|y-z\|^2 + \cancel{+ \|x-(y-z)\|^2} \\ &= 2\|x+z\|^2 + 2\|y\|^2 - 2\|x\|^2 - 2\|y-z\|^2 \\ &= 2(2\|x\|^2 + 2\|z\|^2 - \|x-z\|^2) + 2\|y\|^2 - 2\|x\|^2 - 2\|y-z\|^2 \\ &= 2\|x\|^2 + 4\|z\|^2 - 2\|x-z\|^2 + 2\|y\|^2 - 2\|y-z\|^2 \\ &= (2\|x\|^2 + 2\|z\|^2 - 2\|x-z\|^2) + (2\|y\|^2 + 2\|z\|^2 - 2\|y-z\|^2) \\ &= 4g(x, z) + 4g(y, z) \end{aligned}$$

$$\therefore \underline{g(x+y, z) = g(x, z) + g(y, z)} \quad \forall x, y, z \in V.$$

(iv.) Show  $g(kx, y) = k g(x, y) \quad \forall k \in \mathbb{N}$

$k=1$  is immediate.

For  $k=2$ ,  $g(2x, y) = g(x+x, y) = g(xy) + g(x, y) = 2g(x, y)$ ,

where I used (iii.). Suppose  $g(kx, y) = k g(x, y)$  for some  $k \in \mathbb{N}$ . Consider,

$$\begin{aligned} g((k+1)x, y) &= g(kx+x, y) \\ &= g(kx, y) + g(x, y) : \text{by (iii.)} \\ &= k g(x, y) + g(x, y) : \text{by induction hypothesis.} \\ &= (k+1) g(x, y) \end{aligned}$$

Hence (iv.) for  $k \Rightarrow$  (iv.) for  $k+1 \therefore$  (iv.) true  $\forall k \in \mathbb{N}$   
by proof by mathematical induction.

(v.) Show  $g(zx, y) = z g(x, y) \quad \forall z \in \mathbb{Z}$ .

$$g(x-x, y) = g(x, y) + g(-x, y) \Rightarrow g(0, y) = g(x, y) + g(-x, y).$$

$$\text{But, } g(0, y) = \frac{1}{4}(\|0+y\|^2 - \|0-y\|^2) = \frac{1}{4}(\|y\|^2 - \|y\|^2) = 0.$$

Thus,  $\underline{g(-x, y) = -g(x, y)} \quad \text{**}$  We have shown  $g(zx, y) = z g(x, y)$  for  $z=0$  and  $z \in \mathbb{N}$  in part (iv.). Consider  $z < 0$  and  $z \in \mathbb{Z} \Rightarrow -z > 0 \Rightarrow -z \in \mathbb{N}$ . Thus,

$$g(-zx, y) = -z g(x, y)$$

$$\Rightarrow -g(zx, y) = -z g(x, y) \text{ by ** with } x \rightarrow zx.$$

Thus  $g(zx, y) = z g(x, y) \quad \forall z \in \mathbb{Z}$ .

(Vi.) Show  $g\left(\frac{p}{q}x, y\right) = \frac{p}{q}g(x, y) \quad \forall p, q \in \mathbb{Z} \text{ with } q \neq 0$

Let  $p, q \in \mathbb{Z}$  with  $q \neq 0$ ,

$$\begin{aligned} g\left(\frac{p}{q}x, y\right) &= g\left(p \cdot \frac{1}{q}x, y\right) && : \frac{p}{q} = p \cdot \frac{1}{q} \\ &= p \cdot g\left(\frac{1}{q}x, y\right) && : \text{by (V.)} \\ &= p \cdot \frac{1}{q} \cdot q g\left(\frac{1}{q}x, y\right) && : 1 = \frac{q}{q} \\ &= \frac{p}{q} g\left(q \cdot \frac{1}{q}x, y\right) && : \text{by (V.)} \\ &= \frac{p}{q} g(x, y). \end{aligned}$$

(Vii.) Notice,  $h(x) = g(x, y) = \frac{1}{4} \left( \|x+y\|^2 - \|x-y\|^2 \right)$

Hence  $h$  is the composition of a translation, the square function and  $x \mapsto \|x\|$  all of which are continuous. It follows  $h(x) \rightarrow h(x_0)$  as  $x \rightarrow x_0$ .

for each  $x_0 \in V$ . (Sorry to be lazy here, maybe some of you gave some evidence that  $x \mapsto \|x\|$  is continuous.)

(Viii.) Let  $r \in \mathbb{R}$  then  $\exists$  a sequence of rational numbers  $\frac{p_n}{q_n} \rightarrow r$  as  $n \rightarrow \infty$ . Consider then, for  $x, y \in V$ ,

$$\begin{aligned} g(rx, y) &= h(rx) \\ &= h\left(\lim_{n \rightarrow \infty} \frac{p_n}{q_n} x\right) \\ &= \lim_{n \rightarrow \infty} h\left(\frac{p_n}{q_n} x\right) && : \text{continuity of } h \\ &= \lim_{n \rightarrow \infty} \left(g\left(\frac{p_n}{q_n} x, y\right)\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{p_n}{q_n} g(x, y)\right) && : \text{by (Vi.)} \\ &= r g(x, y). && : \frac{p_n}{q_n} \rightarrow r. \end{aligned}$$

(ix.)  $g(x, ry+z) = g(ry+z, x)$   
 $= r g(y, x) + g(z, x)$   
 $= \underline{r g(x, y)} + \underline{g(x, z)}.$