

Same instructions as Mission 1. Thanks!

**Problem 37** Your signature below indicates you have:

- (a.) I have read much of Cook's Chapter 4 and 5: \_\_\_\_\_.

**Problem 38** Let  $F(x, y) = (x^3 - 3xy^2, 3x^2y - y^3)$  for all  $(x, y) \in \mathbb{R}^2$ . Show  $F$  is locally invertible at all points in the plane except one. Find the inverse for  $F$  restricted to the sector  $-\pi/3 < \theta < \pi/3$  for  $r > 0$  (I use the usual polar coordinates in the plane)

**Problem 39** Let  $F(x, y, z) = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(x^2 - yz, z^2 - xy, y^2 - xz)$ . Find the inverse function of  $F$ , or, if not globally possible, find a local inverse for  $F$ .

**Problem 40** Edwards, page 194, #3.7.

**Problem 41** Let  $G(x, y, a, b) = (x^2 - y^2 - ax + by, 2xy - xb - ya)$ . Suppose  $M = G^{-1}(2, 1)$ .

- (a.) Solve for  $a, b$  as functions of  $x, y$
- (b.) use the implicit function theorem to show where it is possible to solve for  $a, x$  as functions of  $b, y$  (no need to actually solve it, demonstration of existence suffices)
- (c.) use the implicit function theorem to show where it is possible to solve for  $a, y$  as functions of  $b, x$  (no need to actually solve it, demonstration of existence suffices))
- (d.) use the implicit function theorem to show where it is possible to solve for  $x, y$  as functions of  $a, b$ . (no need to actually solve it, demonstration of existence suffices))

*note: I don't expect you to analyze the subtle question of if it is still possible to solve where there implicit function theorem breaks down. I merely wish for you to find the low-hanging fruit which the implicit function theorem provides*

**Problem 42** Consider  $F(x, y, z, w) = (x^2 + y^2, z^2 - w^2)$ . Define  $M = F^{-1}(5, -7)$ . Find the tangent space and normal space to  $M$  at the point  $p = (1, 2, 3, 4)$ .

**Problem 43** Find a parametrization of  $M$  in the previous problem near the given point. Verify the tangent space you found by utilizing the parametrization of  $M$  as appropriate.

**Problem 44** Let  $G(x, y, z, w) = x^2 + y^2$  and consider  $M = G^{-1}(2)$ . Find the tangent space and normal space to  $M$  at the point  $p = (1, 1, 2, 3)$ .

**Problem 45** Suppose  $G_1 : \mathbb{R}^5 \rightarrow \mathbb{R}$  and  $G_2 : \mathbb{R}^5 \rightarrow \mathbb{R}^2$  are smooth functions and  $M_1 = G_1^{-1}\{c_1\}$  and  $M_2 = G_2^{-1}\{(c_2, c_3)\}$  are non-empty manifolds where  $G_1$  and  $G_2$  are both full-rank;  $G'_1(p)$  has rank 1 at each point in  $M_1$  and  $G'_2(q)$  has rank 2 at each point  $q$  in  $M_2$ . If we define<sup>1</sup>  $G = (G_1, G_2)$  then explain how  $M = G^{-1}\{(c_1, c_2, c_3)\}$  relates to  $M_1$  and  $M_2$ .

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<sup>1</sup>I make the identification  $(a, (b, c)) = (a, b, c)$  here

**Problem 46** Let  $S_1$  be the unit-circle and  $S_2$  be the unit-sphere. Find a parametrization of  $M = S_1 \times S_2 \subseteq \mathbb{R}^5$ . Find the tangent space at an arbitrary point in  $M$

**Problem 47** Consider  $M$  in the previous problem. Express  $M$  as the level-set of an appropriate function. Find the normal space to  $M$  at an arbitrary point on  $M$ .

**Problem 48** If  $x^2 + y^2 + z^2 + w^2 = 1$  and  $xywz = 1$  then calculate  $\frac{\partial z}{\partial x}|_y$ . That is, take  $z, w$  to be dependent variables and calculate the derivative of  $z$  with respect to  $x$  while holding  $y$ -fixed.

**Problem 49** Consider  $PV = nRT$  where  $P, V, n, T$  are variables. If  $P = V^2$  then calculate  ~~$\frac{\partial T}{\partial V}$~~  assuming  $T = T(V, P)$ .

**Problem 50** Edwards page 116, #5.6 (Lagrange multipliers)

**Problem 51** Edwards page 116, #5.9 (Lagrange multipliers)

**Problem 52** Edwards page 116, #5.10 (Lagrange multipliers)

**Problem 53**  $\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$  denotes Hamilton's **quaternions**. In particular, a quaternion has the form  $\alpha = t + xi + yj + zk$  for  $t, x, y, z \in \mathbb{R}$  and the **versors**  $i, j, k$  satisfy the relations:  $-ji = ij = k$  and  $-kj = jk = i$  and  $-ik = ki = j$  and  $i^2 = j^2 = k^2 = -1$ . Define the quaternionic conjugate of  $\alpha$  by:

$$\bar{\alpha} = t - xi - yj - zk$$

calculate the formula for  $\alpha\bar{\alpha}$  and use the given notation to define  $S_3$  as a subset of the quaternions. (we discussed  $S_3$  in Lecture, it is the unit 3-sphere which lives in 4-dimensional space)

**Problem 54** Show that  $T_p M$  is indeed a subspace of  $T_p \mathbb{R}^n$  for the case of a parametrized  $M$

(in my notes, I prove the implicitly defined case, but, you might notice proof that the set of all tangent vectors to a parametrized manifold is absent from my current notes. To prove this, you need to show that the sum and scalar multiple of tangent vectors are once again tangent vectors where the primary definition of a tangent vector is given in terms of curves)

See sol<sup>b</sup> for improved problem statement.

### Mission 3 Solution : Advanced Calculus

WARNING: P38, P39, are very difficult unless you have my trick, don't despair if the algebra was impossible here.

P38 Let  $F(x, y) = (x^3 - 3xy^2, 3x^2y - y^3)$ . Consider,

$$F'(x, y) = \begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}$$

$$\begin{aligned} \det(F'(x, y)) &= (3x^2 - 3y^2)^2 + 36x^2y^2 \\ &= 9x^4 - 18x^2y^2 + 9y^4 + 36x^2y^2 \\ &= 9(x^4 + 2x^2y^2 + y^4) \\ &= 9(x^2 + y^2)^2 \neq 0 \quad \forall (x, y) \in \mathbb{R}^2 - \{(0, 0)\}. \end{aligned}$$

Thus, by Inverse Function Thm,  $F$  is locally invertible in some open nbhd of each point in  $\mathbb{R}^2$ , except the origin.

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To find the inverse in Cartesian coordinates amounts to solving:

$$x^3 - 3xy^2 = u$$

$$3x^2y - y^3 = v$$

for  $x, y$  as functions of  $u \neq v$ . That said,

if  $x = r \cos \theta$  and  $y = r \sin \theta$  then notice,

$$x^3 - 3xy^2 = r^3(\cos^3 \theta - 3\cos \theta \sin^2 \theta) = r^3 \cos(3\theta) = u$$

$$3x^2y - y^3 = r^3(3\cos^2 \theta - \sin^3 \theta) = r^3 \sin(3\theta) = v$$

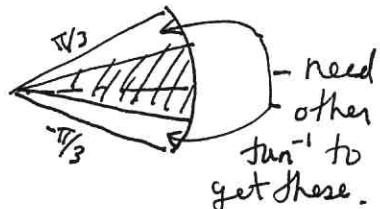
From the above we obtain,

$$\frac{v}{u} = \tan(3\theta) \quad \text{and} \quad u^2 + v^2 = r^6 = (x^2 + y^2)^3$$

$$\tan^{-1}\left(\frac{v}{u}\right) = 3\theta \quad (\text{well, for } -\frac{\pi}{2} < 3\theta < \frac{\pi}{2})$$

$$\theta = \frac{1}{3} \tan^{-1}\left(\frac{v}{u}\right)$$

$$-\frac{\pi}{6} < \theta < \frac{\pi}{6}$$



P38 continued

for  $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$  observe range ( $F$ ) includes all of  $\mathbb{R}^2$  except  $\theta = \pm\pi$ , that is, the negative  $x$ -axis. Range ( $F$ ) =  $\{(u, v) \mid \text{if } v=0 \text{ then } u>0\} = \text{dom}(F^{-1})$ . It remains to exhibit the formula for  $F^{-1}$  (this will establish its existence). Continuing our algebra from the last page, introduce polar coord on  $(u, v)$  by  $u = s \cos \beta$ ,  $v = s \sin \beta \therefore u^2 + v^2 = s^2 \text{ & } \frac{v}{u} = \tan \beta$ .

$$u^2 + v^2 = (x^2 + y^2)^3 = s^2 \therefore s = (x^2 + y^2)^{3/2} \therefore r = s^{1/3}.$$

$$\left. \begin{aligned} \frac{v}{u} &= \tan(3\theta) = \tan(\beta) \\ -\frac{\pi}{3} < \theta < \frac{\pi}{3} &\Rightarrow -\pi < 3\theta < \pi \end{aligned} \right\} \therefore 3\theta = \beta \Rightarrow \theta = \frac{\beta}{3} \text{ where } \theta = \text{Arg}(x, y).$$

Define  $\text{Arg}(x, y)$  to be standard polar angle such that  $-\pi \leq \text{Arg}(x, y) \leq \pi$ . With this notation,

$$F^{-1}(u, v) = (r \cos \theta, r \sin \theta)$$

$$= (s^{1/3} \cos(\frac{\beta}{3}), s^{1/3} \sin(\frac{\beta}{3}))$$

$$= \boxed{(u^2 + v^2)^{1/6} \left( \cos\left(\frac{1}{3}\text{Arg}(u, v)\right), \sin\left(\frac{1}{3}\text{Arg}(u, v)\right) \right)}$$

Notice  $-\pi < \text{Arg}(u, v) < \pi \Rightarrow -\frac{\pi}{3} < \frac{1}{3}\text{Arg}(u, v) < \frac{\pi}{3}$

so Range ( $F^{-1}$ ) does match with the given domain for  $F$ .

P38

Man behind curtain sol<sup>1/2</sup>

$$F(x, y) = (x^3 - 3x^2y, 3xy^2 - y^3)$$

In complex notation,  $(x, y) = x + iy$ ,  $(u, v) = u + iv$  we see

$$\begin{aligned} F(x+iy) &= (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= x^3 - 3xy^2 + i(3x^2y - y^3) \end{aligned}$$

Better yet,

$$F(z) = z^3 \quad \text{so } F'(z) = 3(z^2) \neq 0 \text{ unless } z=0$$

so  $F$  invertible for  $z \neq 0$ .

To find inverse, use complex algebraic techniques,

$$w = z^3$$

$$\Rightarrow z \in w^{1/3}$$

$$\Rightarrow z = \sqrt[3]{w} \quad \text{so } -\frac{\pi}{3} < \operatorname{Arg}(z) < \frac{\pi}{3}.$$

$$\boxed{F^{-1}(w) = \sqrt[3]{w}}$$

Note, from Math 33),  $\sqrt[3]{w} = \sqrt[3]{|w|} \exp\left(\frac{i}{3}\operatorname{Arg}(w)\right)$

$$\text{and } \exp\left(i \frac{\operatorname{Arg}(w)}{3}\right) = \cos\left(\frac{\operatorname{Arg}(w)}{3}\right) + i \sin\left(\frac{\operatorname{Arg}(w)}{3}\right)$$

$$\text{and } |w| = \sqrt{u^2+v^2} \therefore \sqrt[3]{|w|} = (u^2+v^2)^{1/6}.$$

P39 Let  $F(x, y, z) = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (x^2 - yz, z^2 - xy, y^2 - xz)$

Find inverse function for  $F$ , just locally if global inverse not available

Let  $(x, y, z) = x + jy + j^2z$  where  $j^3 = 1$  and

$$(x + jy + j^2z)(a + bj + cj^2) = ax + bxj + cxj^2 + \cancel{a} \\ \cancel{b} + ayj + byj^2 + cyj^3 + \cancel{c} \\ \cancel{c} + azj^2 + bzj^3 + czj^4$$

$$= ax + cy + bz + j(bx + ay + cz) + j^2(cx + by + az)$$

$$= \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Multiplication by  $a + bj + cj^2$  is same as multiplying by  $\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$ .

Consider then,  $w = x + jy + j^2z$  what would  $\frac{1}{w} = ?$

Let  $\frac{1}{w} = a + bj + cj^2$  then we need  $w(\frac{1}{w}) = 1 = (1, 0, 0)$

Thus solve,

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{M(\frac{1}{w})}_{\text{has}} \text{ has } \det(M(\frac{1}{w})) = a^3 + b^3 + c^3 - 3abc. \stackrel{\text{defn}}{=} \nu$$

Cramer's Rule,

$$x = \frac{\det \begin{bmatrix} 1 & c & b \\ 0 & a & c \\ 0 & b & a \end{bmatrix}}{\nu} = \frac{a^2 - bc}{\nu}$$

$$y = \frac{1}{\nu} \det \begin{bmatrix} a & 1 & b \\ b & 0 & c \\ c & 0 & a \end{bmatrix} = \frac{-ab + c^2}{\nu}$$

$$z = \frac{1}{\nu} \det \begin{bmatrix} a & c & 1 \\ b & a & 0 \\ c & b & 0 \end{bmatrix} = \frac{b^2 - ac}{\nu}$$

P 39 continued: I'll check my work to be safe,

$$\begin{aligned}
 & \left[ (a^2 - bc) + j(ab + c^2) + j^2(b^2 - ac) \right] [a + bj + cj^2] = \\
 &= a(a^2 - bc) + b(b^2 - ac) + c(ab + c^2) \\
 &\quad + j((-ab + c^2)a + (b^2 - ac)c + (a^2 - bc)b) \quad \leftarrow \text{reduce to zero.} \\
 &\quad + j^2((b^2 - ac)a + (a^2 - bc)c + (c^2 - ab)b) \\
 &= a^3 + b^3 + c^3 - 3abc \\
 &= \mu
 \end{aligned}$$

Indeed,  $\frac{1}{a + bj + cj^2} = \frac{a^2 - bc + j(c^2 - ab) + j^2(b^2 - ac)}{a^3 + b^3 + c^3 - 3abc}$

Provided  $\mu = (a + b + c)(a^2 + b^2 + c^2 + ab + ac + bc) \neq 0$

It turns out the quadratic term has no real zeros  
hence  $a + b + c = 0$  is the only troublesome case.

$$A^x = (\mathbb{R} \oplus j\mathbb{R} + j^2\mathbb{R})^x = \{a + bj + cj^2 \mid a + b + c \neq 0\}$$

is the group of units for the associative algebra  $A = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ . Well, with this technology in mind,

$$F(x, y, z) = \frac{(x^2 - yz) + j(z^2 - xy) + j^2(y^2 - xz)}{x^3 + y^3 + z^3 - 3xyz} = \frac{1}{x + jy + j^2z}$$

That is  $F(\alpha) = \frac{1}{\alpha}$  to find inverse solve  $\frac{1}{\alpha} = \beta$  for  $\alpha$ .

Simply,  $\alpha = \frac{1}{\beta} \therefore F^{-1}(\beta) = \frac{1}{\beta} = F(\beta)$ . This map is its own inverse. Let  $F(x, y, z) = (a, b, c)$  then,

$$F^{-1}(a, b, c) = \frac{(a^2 - bc, c^2 - ab, b^2 - ac)}{a^3 + b^3 + c^3 - 3abc}$$

P39 continued

$$F'(x, y, z) = \left[ \frac{\partial F}{\partial x} \mid \frac{\partial F}{\partial y} \mid \frac{\partial F}{\partial z} \right]$$

Or,  $F(\alpha) = \frac{1}{\alpha} \Rightarrow F'(\alpha) = -\frac{1}{\alpha^2}$  thus  $F$   
is locally invertible wherever  $\alpha \in A^\times$ .  
( $\alpha = 0$  and zero divisors of  $A$  are  
troublesome for reciprocal powers of  
the algebra variable  $\alpha$ )

In contrast, to use inverse function theorem  
at level of real notation. I'd need to  
calculate: for  $F = \frac{1}{\mu}(x^2 - yz, z^2 - xy, y^2 - xz) = \frac{1}{\mu} G$

$$\det(F') = \det \left[ \begin{array}{ccc} \frac{-1}{\mu^2} \frac{\partial \mu}{\partial x} G + \frac{1}{\mu} \frac{\partial G}{\partial x} & \frac{-1}{\mu^2} \frac{\partial \mu}{\partial y} G + \frac{1}{\mu} \frac{\partial G}{\partial y} & \frac{-1}{\mu^2} \frac{\partial \mu}{\partial z} G + \frac{1}{\mu} \frac{\partial G}{\partial z} \\ \end{array} \right]$$

I refuse. It's awful.

My Point: algebra-based substitutions are very  
useful if you can find them.

**PROBLEM 40**

Edwards #3.7 pg. 194

$$G_1 = xu^2 + yzv + x^2z = 3$$

$$G_2 = xyv^3 + 2zu - u^2v^2 = 2$$

$$G' = \begin{bmatrix} u^2 + 2xz & zv & yv + x^2 \\ yv^3 & xv^3 & zu \\ G_u & G_v & \end{bmatrix}$$

later

Here  $(G_1, G_2) = G : \mathbb{R}_{x,y,z,v}^5 \rightarrow \mathbb{R}^2$ . Can we solve for  $(u, v)$  near  $(1, 1)$  as a function of  $(x, y, z)$  near  $(1, 1, 1)$ ?

$$G' = \begin{bmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} & \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{bmatrix}$$

$$\begin{bmatrix} G_x & G_y & G_z \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} + \begin{bmatrix} G_u & G_v \end{bmatrix} \begin{bmatrix} u-1 \\ v-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

evaluating the derivatives at  $(1, 1, 1, 1)$  gives the linearization of  $G = (3, 2)$  at  $(1, 1, 1, 1, 1)$ . We can solve for  $u, v$  if  $[G_u | G_v]$  is non-singular, observe

$$\begin{bmatrix} G_u & G_v \end{bmatrix} = \begin{bmatrix} 2xu & yz \\ 2z - zuv^2 & 3xyv^2 - zu^2v \end{bmatrix}$$

$$\begin{bmatrix} G_u & G_v \end{bmatrix}(1, 1, 1, 1) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and } \det \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = 2 \neq 0$$

Therefore, as  $G \in C^1(1, 1, 1, 1)$  and the necessary submatrix of  $G'$  is invertible it follows that we may solve for  $u, v$  as functions of  $x, y, z$  as desired. Moreover, the  $L^2$  approx is obtained from:

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3(x-1) + y-1 + 2(z-1) \\ x-1 + y-1 + 2(z-1) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2(x-1) \\ 2(x-1) + 2(y-1) + 4(z-1) \end{bmatrix} = \begin{bmatrix} 1 - (x-1) \\ 1 - (x-1) - (y-1) - 2(z-1) \end{bmatrix} \end{aligned}$$

PROBLEM 40 continued

We find, as 1<sup>st</sup> approximation to  $G = 0$ ,

$$U = 2 - x$$

$$V = -x - y - 2z + 5$$

Solves  $G(x, y, z, u, v) = (3, 2)$  near  $(1, 1, 1, 1, 1)$ .

[P41] Let  $G(x, y, a, b) = (x^2 - y^2 - ax + by, 2xy - xb - ya)$   
and define  $M = G^{-1} \{ (2, 1) \}$ .

(a.) Solve for  $a, b$  as functions of  $x, y$  on  $M$ .

$$\begin{cases} x^2 - y^2 - ax + by = 2 \\ 2xy - xb - ya = 1 \end{cases} \xrightarrow{\text{★}} \begin{bmatrix} -x & y \\ -x & -y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 - x^2 + y^2 \\ 1 - 2xy \end{bmatrix}$$

Multiply by inverse  $\frac{1}{xy+xy} \begin{bmatrix} -y & -y \\ x & -x \end{bmatrix}$  to obtain,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2xy} \begin{bmatrix} -y & -y \\ x & -x \end{bmatrix} \begin{bmatrix} 2 - x^2 + y^2 \\ 1 - 2xy \end{bmatrix}$$

$$\begin{aligned} &= \frac{1}{2xy} \begin{bmatrix} -y(2 - x^2 + y^2) - y(1 - 2xy) \\ x(2 - x^2 + y^2) - x(1 - 2xy) \end{bmatrix} \\ &= \frac{1}{2xy} \begin{bmatrix} -3y + yx^2 - y^3 + 2xy^2 \\ x - x^3 + xy^2 + 2x^2y \end{bmatrix} \end{aligned}$$

Correct  
technique, but  
made error at ★  
I derive correct  
answer  
a rather  
different  
way

Thus,

$$a = \frac{1}{2xy} (-3y + yx^2 - y^3 + 2xy^2)$$

$$b = \frac{1}{2xy} (x - x^3 + xy^2 + 2x^2y)$$

P41 continued: this time, a complex approach,

$$\begin{aligned}x^2 - y^2 - ax + by + i(2xy - xb - ya) &= \\&= (x+iy)^2 + a(-x-iy) + b(y-xi) \\&= (x+iy)^2 - (a+ib)(x+iy)\end{aligned}$$

Let  $\lambda = a+ib$  then  $G(z, \lambda) = z^2 - \lambda z$  and

$M$  is defined as ~~as~~  $(z, \lambda) \in \mathbb{C}^2$  such that  $z^2 - \lambda z = 2+i$

That is,  $\lambda = \frac{z^2 - 2 - i}{z}$ . In other words,  $M$  is a complex - curve, a graph in  $\mathbb{C}^2$  in fact.

$$\begin{aligned}\lambda &= z - \frac{2-i}{z} \\&= x+iy - \frac{(2-i)(x-iy)}{x^2+y^2} \\&= \underbrace{x + \frac{1}{x^2+y^2}(2x-y)}_a + i \underbrace{\left[y + \frac{1}{x^2+y^2}(x+2y)\right]}_b\end{aligned}$$

Remark: don't get the wrong idea from P38 & P41.  
it is not usually the case that a  $\mathbb{C}$ -based substitution  
makes the problem easier to understand. These are special.

P41 continued

$$G(x, y, a, b) = \begin{bmatrix} x^2 - y^2 - ax + by \\ 2xy - xb - ya \end{bmatrix}$$

$$G'(x, y, a, b) = \begin{bmatrix} 2x - a & -2y + b & -x & y \\ 2y - b & 2x - a & -y & -x \end{bmatrix}$$

(b.) to solve for  $a = a(b, y)$  and  $x = x(b, y)$  we

$$\text{need } \det \left[ \frac{\partial G}{\partial x} \middle| \frac{\partial G}{\partial a} \right] = \det \begin{bmatrix} 2x - a & -x \\ 2y - b & -y \end{bmatrix} \neq 0$$

points where,  $-2xy + ay + 2xy - bx$

$$(2x - a)(-y) - (-x)(2y - b) = \boxed{ay - bx \neq 0}$$

(c.) to solve for  $a, y$  as funcs of  $b, x$  we need (to apply implicit func. Th<sup>m</sup> at least) that

$$\begin{aligned} \det \left[ \frac{\partial G}{\partial y} \middle| \frac{\partial G}{\partial a} \right] &= \det \begin{bmatrix} -2y + b & -x \\ 2x - a & -y \end{bmatrix} \\ &= 2y^2 - by + x(2x - a) \\ &= \boxed{2(x^2 + y^2) - ax - by \neq 0} \end{aligned}$$

(d.) to solve  $x = x(a, b), y = y(a, b)$  we would like to have  $\det \left[ \frac{\partial G}{\partial x} \middle| \frac{\partial G}{\partial y} \right] \neq 0$  to apply implicit func. Th<sup>m</sup>,

$$\det \begin{bmatrix} 2x - a & b - 2y \\ 2y - b & 2x - a \end{bmatrix} = \boxed{(2x - a)^2 + (2y - b)^2 \neq 0}$$

(the condition above only fails if both  $2x = a$  and  $2y = b$ )

Bonus comment.

Notice,  $z^2 - 2z = z + i \Rightarrow (z - \frac{1}{2})^2 = z + i - \frac{1}{4}$   
 $\Rightarrow z = \frac{1}{2} \pm \sqrt{z + i - \frac{1}{4}}$

I'd expect for this example, we need both +/- solns near points with  $2x = a$  &  $2y = b$

[P42] Consider  $F(x, y, z, w) = (x^2 + y^2, z^2 - w^2)$

Define  $M = F^{-1}\{(5, 7)\}$ , find tangent & normal space to  $M$  at  $p = (1, 2, 3, 4)$

$$F'(x, y, z, w) = \begin{bmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 2z & -2w \end{bmatrix} = \begin{bmatrix} (\nabla F_1)^T \\ (\nabla F_2)^T \end{bmatrix}$$

$$N_p M = \text{span} \{ (p, \langle 2, 4, 0, 0 \rangle), (p, \langle 0, 0, 6, -8 \rangle) \}$$

The tangent space  $T_p M$  is complementary;  $T_p M \oplus N_p M = T_p \mathbb{R}^4$

I can see it by inspection,

$$T_p M = \text{span} \{ (p, \langle -4, 2, 0, 0 \rangle), (p, \langle 0, 0, 8, -6 \rangle) \}$$

A method to calculate vectors  $\perp$  to a given set is to place their transposes in a matrix and calculate the null-space, since  $\nabla F_1(p) = (2, 4, 0, 0)$  and  $\nabla F_2(p) = (0, 0, 6, -8)$  we find

$$\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 0 & 6 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} \end{bmatrix}$$

If  $(v_1, v_2, v_3, v_4) \in \text{Null}(F'(p))$  then  $v_1 = -2v_2$  &  $v_3 = \frac{4}{3}v_4$

$$\text{hence } (v_1, v_2, v_3, v_4) = (-2v_2, v_2, \frac{4}{3}v_4, v_4)$$

$$= v_2 \underbrace{(-2, 1, 0, 0)}_{\text{these vectors } \perp \text{ to}} + v_4 \underbrace{(0, 0, \frac{4}{3}, 1)}_{\langle 2, 4, 0, 0 \rangle \text{ & } \langle 0, 0, 6, -8 \rangle}$$

Thus, we once again obtain

$$T_p M = \text{span} \{ (p, \langle -2, 1, 0, 0 \rangle), (p, \langle 0, 0, \frac{4}{3}, 1 \rangle) \}$$

Remark: my choice of spanning sets for  $N_p M$  &  $T_p M$  is by no means the only choice.  $\exists$  only many bases for these 2-dim'l subspaces of  $T_p \mathbb{R}^4$ .

[P43] Parametrize M from [P42],

$$x^2 + y^2 = 5 \quad \text{and} \quad z^2 - w^2 = 7$$

$$x = \sqrt{5} \cos \theta \quad z = \sqrt{7} \sinh \beta$$

$$y = \sqrt{5} \sin \theta \quad w = \sqrt{7} \cosh \beta$$

I invite the reader to verify the verity of my parametrization above. In summary, for  $0 \leq \theta \leq 2\pi$ ,  $\beta \in \mathbb{R}$ ,

$$\Phi(\theta, \beta) = (\sqrt{5} \cos \theta, \sqrt{5} \sin \theta, \sqrt{7} \sinh \beta, \sqrt{7} \cosh \beta)$$

$$\frac{\partial \Phi}{\partial \theta} = \langle -\sqrt{5} \sin \theta, \sqrt{5} \cos \theta, 0, 0 \rangle$$

$$\frac{\partial \Phi}{\partial \beta} = \langle 0, 0, \sqrt{7} \cosh \beta, \sqrt{7} \sinh \beta \rangle$$

To study (1, 2, 3, 4) we choose  $\theta_0, \beta_0$  for which

$$1 = \sqrt{5} \cos \theta_0, \quad 2 = \sqrt{5} \sin \theta_0 \quad \text{and} \quad 3 = \sqrt{7} \sinh \beta_0, \quad 4 = \sqrt{7} \cosh \beta_0$$

$$\frac{\partial \Phi}{\partial \theta}(\theta_0, \beta_0) = \langle -2, 1, 0, 0 \rangle$$

$$\frac{\partial \Phi}{\partial \beta}(\theta_0, \beta_0) = \langle 0, 0, 4, 3 \rangle$$

Indeed,  $T_p M = \text{span} \left\{ (p, \langle -2, 1, 0, 0 \rangle), (p, \langle 0, 0, 4, 3 \rangle) \right\}$ .

Remark: for parametrized surface M, we can calculate  $T_p M$  directly from the span of the partial velocities of the parametrization.

Slogan: Normals are natural for implicitly defined manifolds whereas Tangents are natural for parametrically presented manifolds.

P44  $G(x, y, z, w) = x^2 + y^2$ , define  $M = G^{-1}\{2\}$ .

Find  $T_p M$  and  $N_p M$  at  $P = (1, 1, 2, 3)$

$$\nabla G = (2x, 2y, 0, 0) \therefore N_p M = \overline{\text{span}} \left\{ (P, \langle 2, 2, 0, 0 \rangle) \right\}$$

To find tangent space we need  $\langle v_1, v_2, v_3, v_4 \rangle$  for which  $\langle v_1, v_2, v_3, v_4 \rangle \cdot \langle 2, 2, 0, 0 \rangle = 0 \Rightarrow \underbrace{2v_1 + 2v_2 = 0}_{v_1 = -v_2}$ .

$$\begin{aligned} \therefore \langle v_1, v_2, v_3, v_4 \rangle &= \langle -v_2, v_2, v_3, v_4 \rangle \\ &= v_2 \langle -1, 1, 0, 0 \rangle + v_3 \langle 0, 0, 1, 0 \rangle + v_4 \langle 0, 0, 0, 1 \rangle \end{aligned}$$

$$\therefore T_p M = \overline{\text{span}} \left\{ (P, \langle -1, 1, 0, 0 \rangle), (P, \langle 0, 0, 1, 0 \rangle), (P, \langle 0, 0, 0, 1 \rangle) \right\}$$

Alternatively, parametrize  $M$  by

$$\Phi(\theta, z, w) = (\sqrt{2} \cos \theta, \sqrt{2} \sin \theta, z, w)$$

$$\underbrace{\frac{\partial \Phi}{\partial \theta} = \langle -\sqrt{2} \sin \theta, \sqrt{2} \cos \theta, 0, 0 \rangle}_{\theta = \pi/4}, \quad \frac{\partial \Phi}{\partial z} = \langle 0, 0, 1, 0 \rangle, \quad \frac{\partial \Phi}{\partial w} = \langle 0, 0, 0, 1 \rangle$$

$\theta = \pi/4$  at  $P = (1, 1, 2, 3)$  so you see how we get the answer from this approach.

p 45

$$G_1 : \mathbb{R}^5 \rightarrow \mathbb{R}, M_1 = G_1^{-1}\{c_1\}, G_1 \text{ rank 1 on } M_1$$

$$G_2 : \mathbb{R}^5 \rightarrow \mathbb{R}^2, M_2 = G_2^{-1}\{(c_2, c_3)\}, G_2 \text{ rank 2 on } M_2$$

Let  $G = (G_1, G_2)$  and explain how  $M = G^{-1}\{(c_1, c_2, c_3)\}$  relates to  $M_1$  and  $M_2$

If  $P \in M$  then  $G(P) = (G_1(P), G_2(P)) = (c_1, c_2, c_3)$

thus  $G_1(P) = c_1$  and  $G_2(P) = (c_2, c_3)$ . Thus  $P \in G_1^{-1}\{c_1\}$

and  $P \in G_2^{-1}\{(c_2, c_3)\}$  which shows  $P \in M_1$  and  $P \in M_2$

consequently  $P \in M_1 \cap M_2$ . Conversely if  $P \in M_1 \cap M_2$

then  $G_1(P) = c_1$  &  $G_2(P) = (c_2, c_3) \Rightarrow G(P) = (c_1, c_2, c_3)$

and  $P$  is found to be in  $M \therefore M_1 \cap M_2 \subseteq M$ .

We find  $M = M_1 \cap M_2$ .

//

Now, what I didn't quite ask, and is much more subtle, is... is  $M$  a manifold?

Or, what is  $\dim(M)$ ? Some special cases,  $G_2 = (G_2^1, G_2^2)$ ,

1.)  $\{\nabla G_1(P), \nabla G_2^1(P), \nabla G_2^2(P)\}$  are LI  $\forall P \in M$

then  $\text{rank}(G) = 3$  constantly on  $M$  and we deduce  $M$  is two-dim'l.

2.)  $\nabla G_1(P) \in \text{span}\{\nabla G_2^1(P), \nabla G_2^2(P)\} \forall P \in M$

then constant rank of  $G$  is 2 and we deduce  $M$  is three-dim'l.

In principle, you could have  $M$  somewhere 2-dim'l and elsewhere 3-dim'l given our data on  $M_1$  &  $M_2$ .

**P46** Let  $S_1'$  be the unit-circle and  $S_2'$  be the unit-sphere.  
 Find a parametrization of  $M = S_1' \times S_2' \subseteq \mathbb{R}^5$ . Find  
 the  $T_p M$  for an arbitrary point  $p \in M$ .

$$\underline{\Phi}(\theta, \alpha, \beta) = (\cos \theta, \sin \theta, \cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta) \quad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \alpha \leq \pi \end{array}$$

I invite the reader to verify for  $\underline{\Phi}(\theta, \alpha, \beta) = (x_1, x_2, x_3, x_4, x_5)$   
 we indeed have  $x_1^2 + x_2^2 = 1$  &  $x_3^2 + x_4^2 + x_5^2 = 1$ .

$$T_p M = \text{span} \left\{ \left( p, \frac{\partial \underline{\Phi}}{\partial \theta} \right), \left( p, \frac{\partial \underline{\Phi}}{\partial \alpha} \right), \left( p, \frac{\partial \underline{\Phi}}{\partial \beta} \right) \right\}$$

$$\boxed{T_p M = \text{span} \left\{ \left( p, \langle -\sin \theta_0, \cos \theta_0, 0, 0, 0 \rangle \right), \right.} \\ \left. \left( p, \langle 0, 0, -\sin \alpha_0 \sin \beta_0, \cos \alpha_0 \sin \beta_0, 0 \rangle \right), \right. \\ \left. \left( p, \langle 0, 0, \cos \alpha_0 \cos \beta_0, \sin \alpha_0 \cos \beta_0, -\sin \beta_0 \rangle \right) \right\}}$$

Where  $p = \underline{\Phi}(\theta_0, \alpha_0, \beta_0)$ .

**P47**  $M = G^{-1}\{(1,1)\}$ , where  $G(x_1, x_2, x_3, x_4, x_5) = (x_1^2 + x_2^2, x_3^2 + x_4^2 + x_5^2)$ .

$$N_p M = \text{span} \left\{ (p, \nabla G_1(p)), (p, \nabla G_2(p)) \right\}$$

$$\boxed{N_p M = \text{span} \left\{ (p, \langle 2\bar{x}_1, 2\bar{x}_2, 0, 0, 0 \rangle), (p, \langle 0, 0, 2\bar{x}_3, 2\bar{x}_4, 2\bar{x}_5 \rangle) \right\}}$$

where  $p = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$ .

Remark: we can see  $(N_p M)^\perp = T_p M$   
 or  $(T_p M)^\perp = N_p M$

by comparing **P47** & **P48**.

P48 If  $x^2 + y^2 + z^2 + w^2 = 1$  and  $xyzw = 1$   
 Then calculate  $\frac{\partial z}{\partial x}|_y$ .

$$\text{Sol 1/} \quad 2x + 2z \frac{\partial z}{\partial x}|_y + 2w \frac{\partial w}{\partial x}|_y = 0 \Rightarrow \frac{\partial w}{\partial x}|_y = \frac{-2x - 2z \frac{\partial z}{\partial x}|_y}{2w}$$

$$yzw + xy \frac{\partial z}{\partial x}|_y w + xyz \frac{\partial w}{\partial x}|_y = 0$$

$$yzw + xyw \frac{\partial z}{\partial x}|_y + xyz \left( \frac{-2x - 2z \frac{\partial z}{\partial x}|_y}{2w} \right) = 0$$

$$\boxed{\frac{\partial z}{\partial x}|_y = \frac{1}{xyw - xyz \frac{z}{w}} \left( -yzw + \frac{x^2yz}{w} \right)}$$

$$= \left( \frac{zw}{z^2 - w^2} \right) \left( \frac{1}{xy} \left( \frac{x^2y}{w} - yw \right) \right) = \frac{zw}{z^2 - w^2} \left( \frac{x}{w} - \frac{w}{x} \right)$$

just  
checking  
the  
answers  
match  
up.

Sol 2/

$$2x dx + 2y dy + 2z dz + 2w dw = 0$$

$$yzw dx + xzw dy + xyzw dz + xyzw dw = 0$$

$$= \frac{2w}{z^2 - w^2} \left( \frac{x^2 - w^2}{wx} \right)$$

$$= \frac{2(x^2 - w^2)}{x(z^2 - w^2)}$$

Now, solve for differentials of dep. variables;  $dz, dw$

$$2z dz + 2w dw = -2x dx - 2y dy$$

$$xyzw dz + xyzw dw = -yzw dx - xzw dy$$

$$\frac{1}{z} dz + \frac{1}{w} dw = -\frac{1}{x} dx - \frac{1}{y} dy$$

divide by  
 $xyzw$ .

$$\begin{bmatrix} z & w \\ 1/z & 1/w \end{bmatrix} \begin{bmatrix} dz \\ dw \end{bmatrix} = \begin{bmatrix} -x dx - y dy \\ -y/x dx - y/y dy \end{bmatrix}$$

$$\begin{bmatrix} dz \\ dw \end{bmatrix} = \frac{-1}{z/w - w/z} \begin{bmatrix} 1/w & -w \\ -1/z & z \end{bmatrix} \begin{bmatrix} x dx + y dy \\ dx/x + dy/y \end{bmatrix}$$

$$= \frac{-zw}{z^2 - w^2} \left[ \frac{1}{w} (x dx + y dy) - w \left( \frac{dx}{x} + \frac{dy}{y} \right) \right]$$

$$= \frac{-zw}{z^2 - w^2} \left[ -\frac{1}{z} (x dx + y dy) + z \left( \frac{dx}{x} + \frac{dy}{y} \right) \right]$$

$$\rightarrow dz = \frac{-zw}{z^2 - w^2} \left( \left( \frac{x}{w} - \frac{w}{x} \right) dx + \left( \frac{y}{w} - \frac{w}{y} \right) dy \right) \Rightarrow \boxed{\frac{\partial z}{\partial x}|_y = \frac{zw}{z^2 - w^2} \left( \frac{w}{x} - \frac{x}{w} \right)}$$

P49  $PV = nRT$  and  $P = V^2$

where  $P, V, n, T$  are variables.

I asked you to calculate  $\frac{\partial T}{\partial V}$  assuming  $T = T(V, P)$   
 But, this is impossible as  $T$  cannot be expressed as  
 a function of  $V \& P$ . Sorry.

Instead, calculate  $\left. \frac{\partial T}{\partial P} \right|_n$ . Consider,

$$VdP + PdV = RTdn + nRdT$$

$$dP = 2VdV \rightarrow dV = \frac{1}{2V}dP$$

$$\Rightarrow dT = \frac{1}{nR} (VdP + PdV - RTdn)$$

$$= \frac{1}{nR} \left( VdP + \frac{1}{2V}dP - RTdn \right)$$

$$= \underbrace{\frac{1}{nR} \left( V + \frac{1}{2V} \right) dP}_{\left. \frac{\partial T}{\partial P} \right|_n} - \frac{T}{n} dn$$

$\left. \frac{\partial T}{\partial P} \right|_n = \frac{1}{nR} \left( V + \frac{1}{2V} \right)$

$$G(P, V, n, T) = (PV - nRT, P - V^2)$$

$$G'(P, V, n, T) = \begin{bmatrix} V & P & -RT & -nR \\ 1 & -2V & 0 & 0 \end{bmatrix}$$

To say  $T = T(V, P)$  is also to say  $n = n(V, P)$

$$\text{we'd need } \det \left[ \frac{\partial G}{\partial T} \middle| \frac{\partial G}{\partial n} \right] = \det \begin{bmatrix} -nR & -RT \\ 0 & 0 \end{bmatrix} = 0 \neq 0$$

While the implicit function theorem does not tell us what can't be done, this complete failure to meet condition at  $T_h^m$  everywhere seems to spell doom.

**PROBLEM 50**

5.6 / Find dimensions of box of maximal volume which can be inscribed in  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ .

Objective function:  $V = 8xyz$  (draw picture to see this constraint:  $g(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0$ )

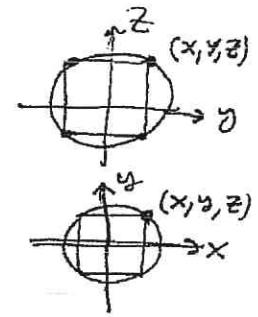
$$\nabla V = \lambda \nabla g$$

$$\langle 8yz, 8xz, 8xy \rangle = \lambda \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

$$\begin{aligned} 8yz &= \frac{2\lambda x}{a^2} \\ 8xz &= \frac{2\lambda y}{b^2} \\ 8xy &= \frac{2\lambda z}{c^2} \end{aligned} \quad \left\{ \begin{aligned} \frac{2\lambda}{8} &= \frac{a^2yz}{x} = \frac{b^2xz}{y} = \frac{c^2xy}{z} \\ \Rightarrow a^2y^2z^2 &= b^2x^2z^2 = c^2x^2y^2 \\ \Rightarrow a^2y^2 &= b^2x^2 \neq b^2z^2 = c^2y^2 \\ \Rightarrow \frac{x^2}{a^2} &= \frac{y^2}{b^2} \neq \frac{z^2}{c^2} = \frac{y^2}{b^2} \\ \Rightarrow \frac{x^2}{a^2} &= \frac{y^2}{b^2} = \frac{z^2}{c^2} \\ \Rightarrow \frac{3x^2}{a^2} &= \frac{3y^2}{b^2} = \frac{3z^2}{c^2} = 1 \end{aligned} \right.$$

$$\Rightarrow x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

$$\therefore V = \frac{8abc}{3\sqrt{3}}$$



(PSI)

Edwards p. 116 # 5.9

Minimize  $f(x) = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$  on  $g(x) = x_1 x_2 \cdots x_n - 1 = 0$   
and thereby deduce the geometric-arithmetic means inequality

Suppose  $x_1, x_2, \dots, x_n > 0$  for what follows. Lagrange multiplier,

$$\nabla f = \lambda \nabla g$$

$$\langle y_n, y_n, \dots, y_n \rangle = \lambda \langle x_2 x_3 \cdots x_n, x_1 x_3 \cdots x_n, \dots, x_1 \cdots x_{n-1} \rangle$$

But  $x_1 x_2 \cdots x_n = 1$  thus

$$\nabla f = \langle y_n, y_n, \dots, y_n \rangle = \lambda \langle y_{x_1}, y_{x_2}, \dots, y_{x_n} \rangle$$

As  $\lambda \neq 0$  we deduce  $\frac{1}{n} = \frac{\lambda}{x_j}$  for  $j=1, 2, \dots, n$  for the extremal sol<sup>n</sup>. Notice,  $x_1 x_2 \cdots x_n = 1$  is not compact, however, we can focus on bounded region and argue the sol<sup>n</sup> to  $\nabla f = \lambda \nabla g$  represents a minimum for  $f$ . As we change the bounding region the max gets bigger and  $\nexists$  an absolute max.

$$\lambda = \frac{x_1}{n} = \frac{x_2}{n} = \dots = \frac{x_n}{n} \Rightarrow x_1 = x_2 = \dots = x_n$$

Hence as  $\begin{cases} x_1 x_2 \cdots x_n = 1 \\ x_j > 0 \forall j \end{cases}$  we obtain  $x_1 = 1, \dots, x_n = 1$ .

$$\text{Thus } f_{\min} = \frac{1}{n}(1+1+\dots+1) = 1 \leq \frac{1}{n}(x_1 + \dots + x_n)$$

$$\Rightarrow 1 = x_1 x_2 \cdots x_n \leq \frac{1}{n}(x_1 + \dots + x_n)$$

$$\Rightarrow \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{1}{n}(x_1 + x_2 + \dots + x_n)$$

If  $x_1, x_2, \dots, x_n > 0$  and  $x_1 x_2 \cdots x_n = c$  then

$$\frac{1}{c} x_1 x_2 \cdots x_n = 1 \Rightarrow \sqrt[n]{\frac{1}{c} x_1 x_2 \cdots x_n} \leq \frac{1}{n+1} \left( \frac{1}{c} + x_1 + \dots + x_n \right)$$

$$\xrightarrow{\text{JUMP}} \Rightarrow \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{1}{n}(x_1 + \dots + x_n)$$

Remark: There is likely something missing here ☺.

PROBLEM 52] problems 5.10, ... of p. 116

$$\begin{array}{l} \text{5.10) Constraints } g_1(x, y, z) = x + 2y + z - 4 \\ g_2(x, y, z) = 3x + y + 2z - 3 \end{array} \quad \left. \begin{array}{l} \{ \\ \} \end{array} \right\} G = (g_1, g_2) = 0$$

gives line of intersection of planes  $g_1 = 0$  &  $g_2 = 0$ .

$$\text{Objective: } f(x, y, z) = x^2 + y^2 + z^2$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\langle 2x, 2y, 2z \rangle = \lambda_1 \langle 1, 2, +1 \rangle + \lambda_2 \langle 3, 1, 2 \rangle$$

$$\begin{aligned} 2x &= \lambda_1 + 3\lambda_2 \Rightarrow 4x = 2\lambda_1 + 6\lambda_2 \rightarrow 4x - 2y = 5\lambda_2 \\ 2y &= 2\lambda_1 + \lambda_2 \quad \quad \quad 2y = 2\lambda_1 + \lambda_2 \\ 2z &= \lambda_1 + 2\lambda_2 \quad \quad \quad 4y = 4\lambda_1 + 2\lambda_2 \rightarrow 4y - 2z = 3\lambda_1 \end{aligned}$$

$$\begin{aligned} 2x &= \lambda_1 + 3\lambda_2 \quad \rightarrow 5\lambda_1 = 6y - 2x \\ 6y &= 6\lambda_1 + 3\lambda_2 \end{aligned}$$

$$\lambda_1 = \frac{6y - 2x}{5} = \frac{4y - 2z}{3} \Rightarrow \underline{18y - 6x = 20y - 10z} \quad \textcircled{I}$$

Note

$$\begin{aligned} 2y &= 2\lambda_1 + \lambda_2 \rightarrow 3\lambda_2 = 4z - 2y \rightarrow \lambda_2 = \frac{4z - 2y}{3} \\ 4z &= 2\lambda_1 + 4\lambda_2 \end{aligned}$$

$$\lambda_2 = \frac{4z - 2y}{3} = \frac{4x - 2y}{5} \rightarrow \underline{20z - 10y = 12x - 6y} \quad \textcircled{II}$$

$$\text{Then } \textcircled{I} \rightarrow 2y = 10z - 6x \\ \underline{y = 5z - 3x}.$$

$$\text{Likewise } \textcircled{II} \rightarrow 4y = 20z - 12x \rightarrow \underline{y = 5z - 3x}.$$

$$\text{Then } g_1 = 0 = x + 2y + z - 4$$

$$g_2 = 0 = 3x + y + 2z - 3$$

$$0 = 3x + (5z - 3x) + 2z - 3 \Rightarrow 7z = 3 \Rightarrow \underline{z = 3/7}.$$

$$\text{But, } z = 3/7 \Rightarrow y = 15/7 - 3x$$

$$g_1 = 0 \rightarrow x + 2\left(\frac{15}{7} - 3x\right) + \frac{3}{7} = 4$$

$$-5x + \frac{30}{7} + \frac{3}{7} = \frac{28}{7} \rightarrow 5x = \frac{5}{7} \therefore \underline{x = 1/7}.$$

$$\text{Then } y = \frac{15}{7} - \frac{3}{7} = \frac{12}{7} = y. \quad \therefore \boxed{\left( \frac{1}{7}, \frac{12}{7}, \frac{3}{7} \right)}$$

P53 Let  $\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$  denote Hamilton's Quaternions.

$$\alpha \in \mathbb{H} \Rightarrow \alpha = t + xi + yj + zk \text{ for } t, x, y, z \in \mathbb{R}$$

and versors  $i, j, k$  satisfy  $i^2 = j^2 = k^2 = -1$  and  $ij = k, jk = i, ki = j$  and  $ji = -k, kj = -i, ik = -j$ . Let the conjugate of  $\alpha$  be

$$\bar{\alpha} = t - xi - yj - zk$$

(calculate  $\alpha\bar{\alpha}$  and define  $S_3$  in this notation.)

$$\alpha\bar{\alpha} = (t + xi + yj + zk)(t - xi - yj - zk)$$

$$\begin{aligned} &= t^2 + (xt)i + (xt)j + (zt)k + \cancel{z} \\ &\quad - (xt)i - \cancel{x^2 i^2} - (xy)ji - (xz)ki + \cancel{z} \\ &\quad - (yt)j - \cancel{y^2 j^2} - (yz)kj + \cancel{z} \\ &\quad - (zt)k - (xz)ik - (yz)jh - \cancel{z^2 k^2}. \end{aligned}$$

everything  
cancels except  
 $t^2, x^2, y^2, z^2$ .

$$\therefore \alpha\bar{\alpha} = t^2 + x^2 + y^2 + z^2$$

Thus,  $S_3 \subseteq \mathbb{H}$  is described as

$$S_3 = \{ \alpha \in \mathbb{H} \mid \alpha\bar{\alpha} = 1 \}$$

P54 Show  $T_p M \subseteq T_p \mathbb{R}^n$  given  $\Phi: D \subseteq \mathbb{R}^n \rightarrow M \subseteq \mathbb{R}^n$   
is a parametrization of  $M$ .

$$\text{Defn } T_p M = \{ (p, \gamma'(0)) \mid \gamma: \mathbb{R} \rightarrow M \text{ with } \gamma(0) = p \}$$

We define addition and scalar multiplication as we did in  $T_p \mathbb{R}^n$ . That is, the  $p$  rides along;

$$(p, v) + (p, w) = (p, v+w)$$

$$c(p, v) = (p, cv).$$

Since  $T_p \mathbb{R}^n$  is a vector space it suffices to show  $T_p M$  is non-empty and closed under vector addition and scalar multiplication.

P54

Let  $P \in M$  and  $u_0 \in D$  for which  $\Phi(u_0) = P$ .

Let  $\gamma: \mathbb{R} \rightarrow M$  be defined by  $\gamma(t) = P \ \forall t \in \mathbb{R}$ .

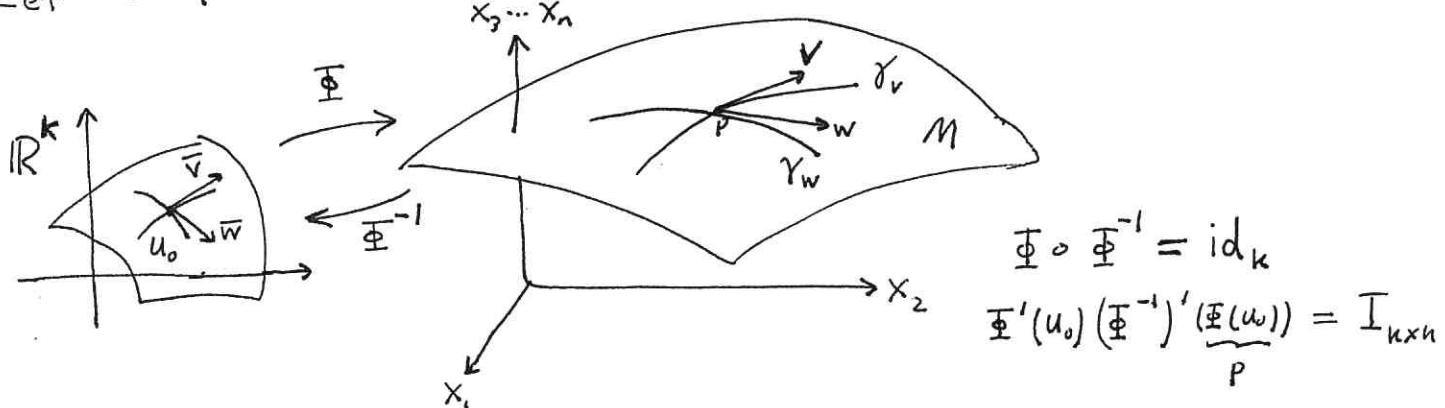
Note  $\gamma'(0) = 0 \therefore (P, 0) \in T_p M$  and we find  $T_p M \neq \emptyset$ .

In fact,  $(P, 0)$  serves as the additive identity in  $T_p \mathbb{R}^n$ .

If  $(P, v), (P, w) \in T_p M$  then  $\exists \gamma_v, \gamma_w: \mathbb{R} \rightarrow M$

for which  $\gamma_v(0) = \gamma_w(0) = P$  and  $\gamma_v'(0) = v, \gamma_w'(0) = w$ .

Let us picture the situation,



$$\Phi \circ \Phi^{-1} = \text{id}_k$$

$$\Phi'(u_0)(\Phi^{-1})'(\underbrace{\Phi(w)}_{P}) = I_{k \times n}$$

Let  $\bar{\gamma}_v = \Phi^{-1} \circ \gamma_v$  and  $\bar{\gamma}_w = \Phi^{-1} \circ \gamma_w$

If  $(\Phi^{-1})'(P) = M$  then the chain-rule yields,

$$\bar{\gamma}'_v(0) = (\Phi^{-1})'(\gamma_v(0)) \gamma_v'(0) = Mv$$

$$\bar{\gamma}'_w(0) = (\Phi^{-1})'(\gamma_w(0)) \gamma_w'(0) = Mw$$

Define  $\bar{v} = Mv$  and  $\bar{w} = Mw$ . Let  $\psi: \mathbb{R} \rightarrow M$  be defined by,

$$\psi(t) = \Phi(u_0 + t(c\bar{v} + \bar{w}))$$

Observe  $\psi(0) = \Phi(u_0) = P$ . Moreover, by chain-rule,

$$\begin{aligned} \psi'(0) &= \frac{d}{dt} [\Phi(u_0 + t(c\bar{v} + \bar{w}))] \Big|_{t=0} \\ &= \Phi'(u_0) (c\bar{v} + \bar{w}) = c \underbrace{\Phi'(u_0)(\Phi^{-1})'(P)}_{I_{k \times k}} v + \underbrace{\Phi'(u_0)(\Phi^{-1})'(P)}_{I_{k \times k}} w \end{aligned}$$

That is,  $\psi'(0) = cv + w \therefore (P, cv + w) \in T_p M$  for all  $c \in \mathbb{R}$  and we've shown  $T_p M$  is closed under + and scalar multiplication.