

Same instructions as Mission 1. Thanks!

Problem 55 Your signature below indicates you have:

(a.) I have read much of Cook's Chapter 6 and 8: _____.

Problem 56 Let $Q(x, y) = 10x^2 - 8xy + 10y^2$. Find a linear change of coordinates for which $Q(\bar{x}, \bar{y}) = \lambda_1\bar{x}^2 + \lambda_2\bar{y}^2$.

Problem 57 Let $Q(x, y, z) = 31x^2 + 15y^2 + 15z^2 - 22xy - 22xz + 10yz$. Find the matrix for Q and use technology to calculate the eigenvalues for Q . Let y_1, y_2, y_3 denote the eigencoordinates, write the formula for Q in terms of the eigencoordinates. For the record, I did not design this problem to have that eigenvalue. It just happened.

Problem 58 Let $Q(x, y, z) = 6x^2 - 2xy - 4xz + 5y^2 - 2yz + 6z^2$. Find the matrix for Q and use technology to calculate the eigenvalues for Q . Let y_1, y_2, y_3 denote the eigencoordinates, write the formula for Q in terms of the eigencoordinates.

Problem 59 I'll share a little backwards linear algebra with you: if v_1, v_2, v_3 are orthonormal vectors then we can build a symmetric matrix by $A = \lambda_1v_1v_1^T + \lambda_2v_2v_2^T + \lambda_3v_3v_3^T$. Let,

$$v_1 = \frac{1}{\sqrt{3}}(1, 1, 1), \quad v_2 = \frac{1}{\sqrt{6}}(1, -2, 1), \quad v_3 = \frac{1}{\sqrt{2}}(1, 0, -1).$$

and $\lambda_1 = 3, \lambda_2 = 6$ and $\lambda_3 = 8$. Calculate A and verify $Av_1 = 3v_1$ and $Av_2 = 6v_2$ and $Av_3 = 8v_3$.

Problem 60 Consider $f(x, y) = 3 + 3(x - 1) + 4(y + 2) + 2(x - 1)^2 + 6(x - 1)(y + 2) + 7(y + 2)^2 + \dots$. Find the values of $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ at the point $(1, -2)$.

Problem 61 Calculate the multivariate Taylor series centered about $(0, 0, 0)$ for $f(x, y, z) = \frac{1+y^2}{1-2xz}$ to order 4. Analyze: is $(0, 0, 0)$ a critical point? If so, analyze if it yields a min/max/saddle or if the second derivative test is not applicable for the given problem.

Problem 62 Find and classify the critical points of $f(x, y) = (x^2 + y^2)e^{x^2-y^2}$.

Problem 63 Consider $\int \cos(tx)dx = \frac{1}{t}\sin(tx) + C$. Use differentiation with respect to t to derive a new integral or two from the given integral. Set $t = 1$ or some other fun value once you've completed the differentiation.

Problem 64 Follow §4 of Professor Conrad's blurb to derive $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$

Problem 65 Read §9 of Professor Conrad's blurb and summarize his argument in your own words.

Problem 66 Find the geodesics on the cone $\phi = \pi/3$ where ϕ denotes the usual spherical angle.

Problem 67 Find a differential equation whose solutions are geodesics on the surface $x^2 + y^3 + z^4 = 1$. Bonus: numerically generate a particular solution and plot it on the surface using some C.A.S.

Problem 68 Let $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$ denote the Lagrangian of a particle with mass m under the force of a spring with potential energy $U(x, y) = \frac{k}{2}(x^2 + y^2)$. Notice $L = T - U$ where T is the kinetic energy. Calculate the Euler-Lagrange equations and show energy $E = T + U$ is conserved along the solution to the Euler-Lagrange equation

Problem 69 Show that the geodesics to a sphere are the great circles.

Problem 70 Let $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + g(r)$ where g is a differentiable function of the polar radius r . Find the Euler Lagrange equations. Also, suppose we define angular momentum $J = \frac{\partial L}{\partial \dot{\theta}}$, show J is conserved.

Problem 71 A marble slides without friction on a bowl of radius R . If the marble has mass m and the force of gravity is given by $-mg\hat{z}$ then find the equations of motion for the marble (differential equations suffice as an answer here).

Problem 72 Imagine a pendulum of length l_1 which consists of a very light rod which does not flex and a bob of mass m_1 . Next, a second pendulum of length l_2 which consists of a very light rod which does not flex and a bob of mass m_2 is attached so that l_2 hangs freely off m_1 . All of this is attached to point and allowed to swing back and forth under the influence of gravity. Assume this is near the surface of the earth where $F = mg$ applies. Find the equations of motion for this double pendulum. Let θ_1 and θ_2 be the angles which l_2 and l_1 make with respect to $-\hat{z}$. Write the equations of motion in terms of these angular variables.

Mission 4 Solution: Advanced Calculus:

P56 Let $Q(x, y) = 10x^2 - 8xy + 10y^2$. Find linear change of coordinates for which $Q(\bar{x}, \bar{y}) = \lambda_1 \bar{x}^2 + \lambda_2 \bar{y}^2$.

Eigen vector/values provide the needed directions & values.

$$A = [Q] = \begin{bmatrix} 10 & -8 \\ -8 & 10 \end{bmatrix} \quad \therefore \det(Q - \lambda I) = \det \begin{bmatrix} 10-\lambda & -8 \\ -8 & 10-\lambda \end{bmatrix}$$

$$= (\lambda - 10)^2 - 64$$

$$= (\lambda - 14)(\lambda - 6)$$

$$\Rightarrow \lambda_1 = 6, \lambda_2 = 14$$

$$(A - 6I)\vec{u}_1 = \begin{bmatrix} 4 & -8 \\ -8 & 4 \end{bmatrix} \vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \hat{u}_1 = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle.$$

$$(A - 14I)\vec{u}_2 = \begin{bmatrix} -4 & -8 \\ -8 & -4 \end{bmatrix} \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \hat{u}_2 = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle.$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \left[\begin{array}{c|c} \hat{u}_1 & \hat{u}_2 \end{array} \right]^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{\text{orthogonal matrix } P^{-1} = P^T}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{x-y}{\sqrt{2}} \end{bmatrix}$$

coordinate change,
 $\Phi_P(v) = [P]^{-1}v$
 for \mathbb{R}^n only!

orthogonal
 matrix $P^{-1} = P^T$,

$$\text{Let } \bar{x} = \frac{1}{\sqrt{2}}(x+y) \quad \text{and} \quad \bar{y} = \frac{1}{\sqrt{2}}(x-y)$$

Observe $Q(\bar{x}, \bar{y}) = 6\bar{x}^2 + 14\bar{y}^2$

[I'm using a theorem about eigenvalues/vectors of symmetric real matrices to guide my work (spectral theorem)]

Let's check it,

$$\begin{aligned} 6\bar{x}^2 + 14\bar{y}^2 &= 3(x+y)^2 + 7(x-y)^2 \\ &= 3x^2 + 6xy + 3y^2 + 7x^2 - 14xy + 7y^2 \\ &= \underline{10x^2 - 8xy + 10y^2}. \end{aligned}$$

PS7 $Q(x, y, z) = 3x^2 + 15y^2 + 15z^2 - 22xy - 22xz + 10yz$

find matrix of Q and use tech. to calc. e-values. Write f-la in eigencoordinates.

$$[Q] = \begin{bmatrix} 3 & -11 & -11 \\ -11 & 15 & 5 \\ -11 & 5 & 15 \end{bmatrix}$$

symmetric $[Q]^T = [Q]$

real eigenvalues

paired with orthonormal e-basis.

Technology will show:

I used for PS7 & PS8
www.arnoldt-bruenner.de
 (see my website
 for link)

$$\lambda_1 = 9$$

$$\lambda_2 = 10$$

$$\lambda_3 = 42$$

$$\therefore Q(\bar{x}, \bar{y}, \bar{z}) = 9\bar{x}^2 + 10\bar{y}^2 + 42\bar{z}^2$$

oops, read direction,

$$Q(y_1, y_2, y_3) = 9y_1^2 + 10y_2^2 + 42y_3^2$$

PS8 Let $Q(x, y, z) = 6x^2 - 2xy - 4xz + 5y^2 - 2yz + 6z^2$
 once more, find matrix for Q and use tech. to find
 e-values & write formula for Q in e-coordinates.

$$[Q] = \begin{bmatrix} 6 & -1 & -2 \\ -1 & 5 & -1 \\ -2 & -1 & 6 \end{bmatrix}$$

$$\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 8$$

$$\therefore Q(\bar{x}, \bar{y}, \bar{z}) = 3\bar{x}^2 + 6\bar{y}^2 + 8\bar{z}^2$$

oops, read direction

$$Q(y_1, y_2, y_3) = 3y_1^2 + 6y_2^2 + 8y_3^2$$

PS9 Let v_1, v_2, v_3 be
 the orthonormal vectors

$$v_1 = \frac{1}{\sqrt{3}}(1, 1, 1), \quad v_2 = \frac{1}{\sqrt{6}}(1, -2, 1), \quad v_3 = \frac{1}{\sqrt{2}}(1, 0, -1)$$

and $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 8$. Let $A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T$.

Calculate A and verify $A v_1 = 3v_1, A v_2 = 6v_2, A v_3 = 8v_3$

I'll have on next page, but for now consider, $v_i^T v_j = v_i \cdot v_j$.

$$A v_1 = \lambda_1 v_1 (v_1^T v_1) + \lambda_2 v_2 (v_2^T v_1) + \lambda_3 v_3 (v_3^T v_1) = \lambda_1 v_1 = \delta_{ij}$$

$$A v_2 = \lambda_1 v_1 (v_1^T v_2) + \lambda_2 v_2 (v_2^T v_2) + \lambda_3 v_3 (v_3^T v_2) = \lambda_2 v_2$$

$$A v_3 = \lambda_1 v_1 (v_1^T v_3) + \lambda_2 v_2 (v_2^T v_3) + \lambda_3 v_3 (v_3^T v_3) = \lambda_3 v_3$$

this is why the construction works in view
 of the orthonormality of v_1, v_2, v_3 .

PS9

Explicitly,

$$\lambda_1 V_1 V_1^T = 3 \left(\frac{1}{\sqrt{3}}\right)^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{\textcircled{I}}$$

$$\lambda_2 V_2 V_2^T = 6 \left(\frac{1}{\sqrt{6}}\right)^2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}}_{\textcircled{II}}$$

$$\lambda_3 V_3 V_3^T = 8 \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 4 \end{bmatrix}}_{\textcircled{III}}$$

You might appreciate $V_2, V_3 \in \text{Null } \textcircled{I}$

and $V_1, V_3 \in \text{Null } \textcircled{II}$ and $V_2, V_1 \in \text{Null } \textcircled{III}$.

Anyway, summing $\textcircled{I}, \textcircled{II}$ and \textcircled{III} yields

$$A = \boxed{\begin{bmatrix} 6 & -1 & -2 \\ -1 & 5 & -1 \\ -2 & -1 & 6 \end{bmatrix}}$$

Then,

$$AV_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 6 & -1 & -2 \\ -1 & 5 & -1 \\ -2 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3V_1.$$

$$AV_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 6 & -1 & -2 \\ -1 & 5 & -1 \\ -2 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 6 \\ -12 \\ 6 \end{bmatrix} = 6 \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 6V_2$$

$$AV_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 6 & -1 & -2 \\ -1 & 5 & -1 \\ -2 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 8 \\ 0 \\ -8 \end{bmatrix} = 8 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 8V_3.$$

Remark: now you know the secret of how to create matrices with nice eigenvalues with an orthonormal basis of your choosing.

P60

$$f(x, y) = 3 + 3(x-1) + 4(y+2) + 2(x-1)^2 + 6(x-1)(y+2) + 7(y+2)^2 + \dots$$

Compare to multivariate Taylor

$$f(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{1}{2} f_{xx}(a, b)(x-a)^2 + f_{xy}(a, b)(x-a)(y-b) + \dots$$

To read by comparison of the coefficients,

| |
|---------------------|
| $f(1, 2) = 3$ |
| $f_x(1, 2) = 3$ |
| $f_y(1, 2) = 4$ |
| $f_{xx}(1, 2) = 4$ |
| $f_{xy}(1, 2) = 6$ |
| $f_{yy}(1, 2) = 14$ |

P61 Find multivariate Taylor Expansion about $(0, 0, 0)$

for $f(x, y, z) = \frac{1+y^2}{1-2xz}$ to order 4. Analyze nature of $(0, 0, 0)$ as a critical point (if it is a critical point...is it?)

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots \quad \text{for } |u| < 1 \quad \text{GEOMETRIC SERIES}$$

$$\frac{1}{1-2xz} = 1 + 2xz + (2xz)^2 + \underbrace{(2xz)^3 + (2xz)^4 + \dots}_{\text{too high order 6, 8 ignore part here.}} + \dots$$

$$f(x, y, z) = (1+y^2)(1 + 2xz + 4x^2z^2 + \dots)$$

$$f(x, y, z) = 1 + 2xz + y^2 + 4x^2z^2 - 2xz^2y^2 + \dots \quad (\text{to 4th order})$$

By inspection, $\nabla f(0, 0, 0) = \langle 0, 0, 0 \rangle \therefore (0, 0, 0)$ is critical. Moreover, we have

$$Q = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$\lambda_1 = -1, \lambda_2 = 1$
 ↪ Saddle point
 neither max nor min.

$$\det(Q - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = \lambda^2(1-\lambda) - (1-\lambda) = (1-\lambda)(\lambda^2-1) = -(\lambda-1)^2(\lambda+1)$$

[P63] Notice $\int \cos(tx) dx = \frac{1}{t} \sin(tx) + C$

Differentiate w.r.t. t ,

$$\frac{\partial}{\partial t} \left[\int \cos(tx) dx \right] = \frac{\partial}{\partial t} \left[\frac{1}{t} \sin(tx) + C \right]$$

$$\int \frac{\partial}{\partial t} (\cos(tx)) dx = -\frac{1}{t^2} \sin(tx) + \frac{1}{t} \cos(tx) \cdot x$$

$$\Rightarrow \int x \sin(tx) dx = \frac{1}{t^2} \sin(tx) - \frac{x}{t} \cos(tx) + C_2. \quad \textcircled{I}$$

Differentiate once more, again w.r.t. t : $\frac{\partial}{\partial t} \int (\) dx = \int \frac{\partial}{\partial t} (\) dx$;

$$\int x^2 \cos(tx) dx = \frac{-2}{t^3} \sin(tx) + \frac{x}{t^2} \cos(tx) + \frac{x}{t^2} \cos(tx) + \frac{x^2}{t} \sin(tx) + C_3. \quad \textcircled{II}$$

Evaluate \textcircled{I} and \textcircled{II} at $t=1$ to obtain,

$$\int x \sin(x) dx = \sin(x) - x \cos(x) + C$$

$$\int x^2 \cos(x) dx = -2 \sin(x) + 2x \cos(x) + x^2 \sin(x) + C$$

These alone are neat enough, but, notice

$$\int x \sin(\pi x) dx = \frac{1}{\pi^2} \sin(\pi x) - \frac{x}{\pi} \cos(\pi x) + C_4.$$

and so many many more...

You see $(\frac{\partial}{\partial t})^n$ will generate further integrals!

[P64], [P65] I leave to the master.

P66 Find geodesics on the cone $\phi = \pi/3$

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2 \phi d\theta^2$$

Here $d\phi = 0$ and $\sin(\pi/3) = \sqrt{3}/2$

$$ds^2 = d\rho^2 + \frac{3}{4} \rho^2 d\theta^2$$

Hence, minimize,

$$\mathcal{J} = \int \underbrace{\left(\dot{\rho}^2 + \frac{3}{4} \rho^2 \dot{\theta}^2 \right)}_{\text{Euler-Lagrange Eq's } L} dt$$

Euler-Lagrange Eq's L

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\rho}} \right) = \frac{\partial L}{\partial \rho} \Rightarrow \frac{d}{dt} (2\dot{\rho}) = \frac{3}{2} \rho \dot{\theta}^2$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt} \left(\frac{3}{2} \rho^2 \dot{\theta} \right) = 0$$

Thus $\frac{3}{2} \rho^2 \dot{\theta} = C$ for some constant C .

Consequently $\dot{\theta} = k/\rho^2$ for a constant $k = 2C/3$

$$\frac{d}{dt} (2\dot{\rho}) = \frac{3}{2} \rho \left(\frac{k}{\rho^2} \right)^2$$

$$\frac{d}{dt} (\dot{\rho}) = \frac{3k^2}{4} \frac{1}{\rho^3}$$

$$\text{Let } V = \ddot{\rho} \text{ then } \frac{dV}{dt} = \frac{d\rho}{dt} \frac{dV}{d\rho} = V \frac{dV}{d\rho} = \frac{3k^2}{4} \frac{1}{\rho^3}$$

$$\text{Hence } V dV = \frac{3k^2}{4} \frac{1}{\rho^3} d\rho \therefore \frac{1}{2} V^2 = -\frac{3k^2}{8} \frac{1}{\rho^2} + C_2$$

$$V = \pm \sqrt{2C_2 - \frac{3k^2}{4} \left(\frac{1}{\rho^2} \right)} = \frac{d\rho}{dt}$$

Oh, I can integrate this, then solve $\frac{3}{2} \rho^2 \dot{\theta} = c$ for θ as function of time to get parametric formulas for the geodesic. But, I hope \beth is easier..

P66 continued w/o time

$$ds^2 = d\rho^2 + \frac{3}{4} \rho^2 d\theta^2$$

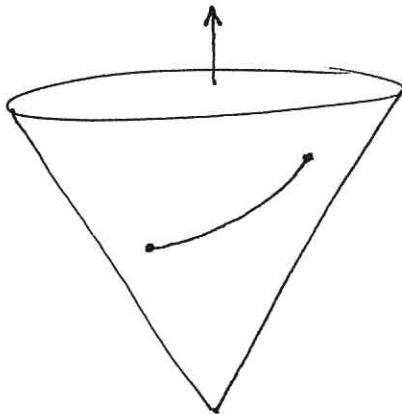
$$\Rightarrow \left(\frac{ds}{d\theta} \right)^2 = \left(\frac{d\rho}{d\theta} \right)^2 + \frac{3}{4} \rho^2.$$

$$L = (\rho')^2 + \frac{3}{4} \rho^2 \quad \text{where } \rho' = \frac{d\rho}{d\theta}$$

$$\frac{\partial L}{\partial \rho} = \frac{d}{d\theta} \left[\frac{\partial L}{\partial \rho'} \right]$$

$$\frac{3}{2} \rho = \frac{d}{d\theta} [2\rho'] \Rightarrow \frac{3}{2} \rho = 2\rho'' \\ \Rightarrow \rho'' - \frac{3}{4} \rho = 0$$

Hence $\boxed{\rho(\theta) = A \cosh \left(\frac{\sqrt{3}}{2} \theta \right) + B \sinh \left(\frac{\sqrt{3}}{2} \theta \right)}.$



Remark: to complete my sol^b show the given
sol^b is obtained from lines on the paper
which is rolled into a cone.

P67 Find DEg² whose sol's are geodesics to $x^2 + y^3 + z^4 = 1$

$$2x\dot{x} + 3y^2\dot{y} + 4z^3\dot{z} = 0$$

$$\dot{x} = \frac{1}{2x}(-3y^2\dot{y} - 4z^3\dot{z})$$

$$\dot{x}^2 = \frac{1}{4x^2}(-3y^2\dot{y} - 4z^3\dot{z})^2 = \frac{1}{4(1-y^3+z^4)}(9y^4\dot{y}^2 - 24y^2z^3\dot{y}\dot{z} + 16z^6\dot{z}^2)$$

$$L = \dot{y}^2 + \dot{z}^2 + \frac{1}{4(1-y^3+z^4)}(9y^4\dot{y}^2 - 24y^2z^3\dot{y}\dot{z} + 16z^6\dot{z}^2)$$

Then, we just need to solve,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}} \quad \text{and} \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z}} \quad \text{for}$$

Sorry, to be explicit, it's quite horrible
I started to write it, ~~but~~ but
so horrible...

Alternatively,

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \lambda(x^2 + y^3 + z^4 - 1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad \therefore$$

$$\ddot{x} = 2\lambda x$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \quad \therefore$$

$$\ddot{y} = 3y^2\lambda$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z} \quad \therefore$$

$$\ddot{z} = 4z^3\lambda$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \lambda}\right) = \frac{\partial L}{\partial \lambda} \quad \therefore$$

$$0 = x^2 + y^3 + z^4 - 1$$

Then, if I was to solve this, I'd try to manipulate:

$$\lambda = \frac{\ddot{x}}{2x} = \frac{\ddot{y}}{3y^2} = \frac{\ddot{z}}{4z^3}$$

P68

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = -kx \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) &= \frac{\partial L}{\partial y} \Rightarrow m\ddot{y} = -ky \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Eq's of Motion}$$

Consider,

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \left[\frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{k}{2}(x^2 + y^2) \right] \\ &= m\dot{x}\ddot{x} + m\dot{y}\ddot{y} + kx\dot{x} + ky\dot{y} \quad \begin{array}{l} kx = -m\ddot{x} \\ ky = -m\ddot{y} \end{array} \\ &= m\dot{x}\ddot{x} - m\ddot{x}\dot{x} + m\dot{y}\ddot{y} - m\ddot{y}\dot{y} \\ &= 0. \end{aligned}$$

Since $\frac{dE}{dt} = 0$ it follows E is conserved (by defⁿ)

P69

Suppose $S = \int \sqrt{R^2\dot{\phi}^2 + R^2\sin^2\phi\dot{\theta}^2} dt$ describes
arc length ... on sphere for curve
parametrized by parameter t . If we take $t = \phi$
then $\phi = t$ and we obtain,

$$S = \int \underbrace{\sqrt{R^2 + R^2\sin^2\phi \left(\frac{d\theta}{d\phi}\right)^2}}_{L} d\phi$$

$$L = R \sqrt{1 + \sin^2\phi \left(\frac{d\theta}{d\phi}\right)^2}$$

(This R will cancel in eq^{ns} below)

$$\frac{d}{d\phi}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \quad \therefore$$

$$\Rightarrow \frac{d}{d\phi}\left(\frac{\sin^2\phi \left(\frac{d\theta}{d\phi}\right)}{\sqrt{1 + \sin^2\phi \left(\frac{d\theta}{d\phi}\right)^2}}\right) = 0 \quad \therefore \frac{(\sin^2\phi)\theta'}{\sqrt{1 + \sin^2\phi (\theta')^2}} = C$$

P 69 continued: set $x = \phi$, $y = \theta$

$$\sin^2(x) \frac{dy}{dx} = c \sqrt{1 + \sin^2(x)} \left(\frac{dy}{dx} \right)^2$$

$$\left(\sin^2 x \frac{dy}{dx} \right)^2 = c^2 \left[1 + \sin^2(x) \left(\frac{dy}{dx} \right)^2 \right]$$

$$(\sin^4 x - c^2 \sin^2 x) \left(\frac{dy}{dx} \right)^2 = c^2$$

$$\left(\frac{dy}{dx} \right)^2 = \frac{c^2}{\sin^2(x) [\sin^2 x - c^2]}$$

$$y = \int \frac{\pm |c| dx}{\sqrt{\sin^2(x) [\sin^2 x - c^2]}}$$

You can probably improve on this answer \circlearrowright

$$\theta = \int \frac{c d\phi}{\sqrt{\sin^4 \phi - c^2 \sin^2 \phi}}.$$

Remark: another approach that would be wise, find the equation of an arbitrary great circle then show it solves our Eq's.

P70

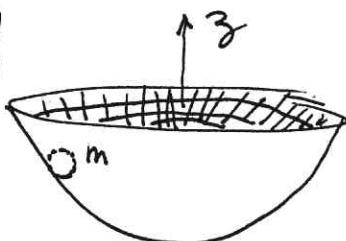
$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + g(r)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow m\ddot{r} = mr\dot{\theta}^2 + \frac{dg}{dr} \quad \text{Euler-Lagrange Eq. } \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0 \Rightarrow \frac{dJ}{dt} = 0$$

(cons. of ang. mom.
is the const. $E_L - E_g = 0$)

P71



$$x^2 + y^2 + z^2 = R^2$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz + \lambda(x^2 + y^2 + z^2 - R^2)$$

Using Lagrange Multiplier λ to impose constraint \rightarrow ,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = 2\lambda x.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \Rightarrow m\ddot{y} = 2\lambda y.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z} \Rightarrow m\ddot{z} = mg + 2\lambda z.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \lambda}\right) = \frac{\partial L}{\partial \lambda} \Rightarrow 0 = x^2 + y^2 + z^2 - R^2$$

In spherical coordinates $z = \rho \cos \phi$

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2 \sin^2 \phi \dot{\theta}^2 + \rho^2 \dot{\phi}^2) + mg\rho \cos \phi + \lambda(\rho - R)$$

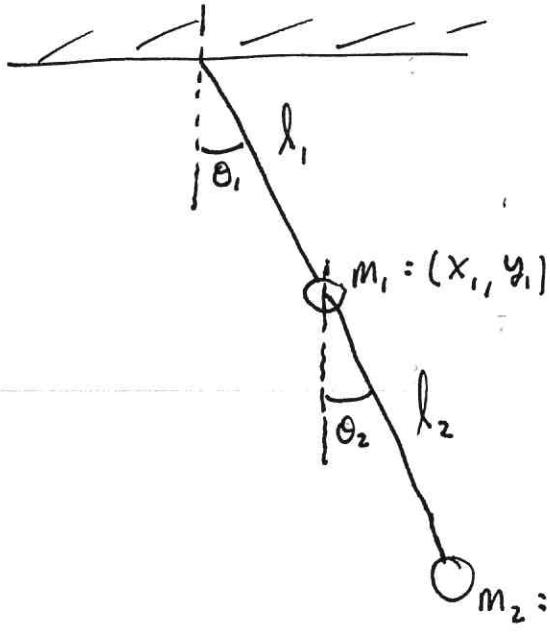
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) = \frac{\partial L}{\partial \rho} \Rightarrow m\ddot{\rho} = m\rho \sin^2 \phi \dot{\theta}^2 + m\rho \dot{\phi}^2 + mg \cos \phi + \lambda.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}(m\rho^2 \sin \phi \dot{\theta}) = 0.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \Rightarrow \frac{d}{dt}(m\rho^2 \dot{\phi}) = -mg \rho \sin \phi.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \lambda}\right) = \frac{\partial L}{\partial \lambda} \Rightarrow 0 = \rho - R.$$

P72



$$X_1 = l_1 \sin \theta_1$$

$$Y_1 = -l_1 \cos \theta_1$$

$$r_1^2 = x_1^2 + y_1^2 = l_1^2$$

$$X_2 = x_1 + l_2 \sin \theta_2$$

$$Y_2 = y_1 - l_2 \cos \theta_2$$

$$x_2^2 + y_2^2 = r_2^2 = l_2^2$$

$$\dot{X}_1^2 + \dot{Y}_1^2 = \dot{r}_1^2 + r_1^2 \dot{\theta}_1^2 \quad \text{oops} - \hat{z} = -\hat{y}$$

$$\dot{X}_2^2 + \dot{Y}_2^2 = \dot{r}_2^2 + r_2^2 \dot{\theta}_2^2$$

$$\begin{aligned} L &= \frac{1}{2}m(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2) - m_1 g y_1 - m_2 g y_2 \\ &= \frac{1}{2}m(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2) - m_1 g(-l_1 \cos \theta_1) - m_2 g(-l_1 \cos \theta_1 - l_2 \cos \theta_2) \\ &= \frac{1}{2}m(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2) + m_1 g l_1 \cos \theta_1 + m_2 g(l_1 \cos \theta_1 + l_2 \cos \theta_2) \end{aligned}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) = \frac{\partial L}{\partial \theta_1} : \quad \underline{\frac{d}{dt} (m l_1^2 \dot{\theta}_1)} = -m_1 g l_1 \sin \theta_1 - m_2 g l_2 \sin \theta_1.$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) = \frac{\partial L}{\partial \theta_2} : \quad \underline{\frac{d}{dt} (m l_2^2 \dot{\theta}_2)} = -m_2 g l_2 \sin \theta_2.$$