Same instructions as Mission 1. Thanks! *What follows is based on an amalgam of my 2015 Lecture Notes Chapter 9-10, Paul Renteln and Jeffrey Lee's texts.*

Problem 91 Your signature below indicates you have:

(a.) I have read much of Cook's Chapter 10: _____.

(b.) I watched enough of Part 3 of Lecture 17 to know what's done there: _____.

(c.) I have read much of Renteln Chapter 1,2,3: _____.

Problem 92 Finish the Example of Part 3 of Lecture 17. Calculate $\star e^1$, $\star e^2$, $\star e^3$ for $\mathbb{R}^3$ with $g = \text{Diag}(-1,-1,1)$. Also, check that the calculations are consistent with some of the Hodge dual properties. In particular check on the $\star\star\,\alpha = (-1)^{\text{index}(g)}(-1)^{n(n-k)}\alpha$ identity.

Problem 93 Let $\text{SL}(2) = \{A \in \mathbb{R}^{2\times 2} \mid \det(A) = 1\}$. Find a coordinate chart which includes the identity matrix.

Problem 94 Exercise 3.25 and 3.26 from page 90 (you can skip the d^2 part, those are zero)

Problem 95 Exercise 3.30 from page 94 (on interior product)

Problem 96 Exercise 3.31 from page 95 (on interior product)

Problem 97 Exercise 3.32 and 3.33 from page 97 (pull-backs)

Problem 98 Exercise 3.34 from page 97 (pull-back)

Problem 99 Exercise 3.40 page 104 (area form on sphere)

Problem 100 Exercise 3.42 page 105 (subspace of tangent space)

Problem 101 Exercise 3.43 page 105-106 (Lie Groups of matrices)

Problem 102 Exercise 3.44 page 106-107 (Lie algebra of matrices)

Problem 103 Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is a linearly **dependent** set of dual-vectors over a finite dimensional vector space V. **Show that** $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k = 0$. For your convenience, you may assume that $\alpha_1 = \sum_{j=2}^{k} c_j \alpha_j$ for some constants $c_2, \ldots, c_k \in \mathbb{R}$.

Problem 104 Suppose γ is a p-form in $\Omega(V)$ where V is a n-dimensional vector space over $\mathbb{R}$. Assume $p \leq n$. If $\gamma \wedge e^j = 0$ for $j = 1, 2, \ldots, n$ then does it follow $\gamma = 0$? Discuss.

Problem 105 Finish the proof of Theorem 10.2.6 part 3. In particular, show: for smooth functions f, g on a manifold $\mathcal{M}$ with coordinates (x^i):

$$\frac{\partial}{\partial x^i}(fg) = \frac{\partial f}{\partial x^i}g + f\frac{\partial g}{\partial x^i}$$

here I omitted the point-dependence for brevity.

Problem 106 Let $F(x, y, z) = (\cos(x), \sin(xy))$. Calculate the push-forward of $X = a\partial_x + b\partial_y + c\partial_z$.

Problem 107 Write the work and flux form mappings in terms of the musical morphisms and Hodge dual with respect to $dx \wedge dy \wedge dz$.

Problem 108 Calculate Hodge duality of forms written in spherical coordinates in $\mathbb{R}^3$. Use this work to derive the formula for the gradient in spherical coordinates.

Mission 6 Sol'n

P92 Calculate $\star e^1$, $\star e^2$, $\star e^3$ for $\mathbb{R}^3$ with $g = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$ and $vol = e^1 \wedge e^2 \wedge e^3$. Note $\epsilon_1 = \epsilon_2 = g(e^1, e^1) = g(e^2, e^2) = -1$, $\epsilon_3 = 1$.

$$\star e^1 = \epsilon_1 e^2 \wedge e^3 = -e^2 \wedge e^3$$
$$\star e^2 = \epsilon_2 e^3 \wedge e^1 = -e^3 \wedge e^1$$
$$\star e^3 = \epsilon_3 e^1 \wedge e^2 = e^1 \wedge e^2$$

Now check $\star\star \alpha = (-1)^{\text{index}(g)} (-1)^{n(n-k)} \alpha$ is identity for k-forms.

$$\star e^1 \wedge e^2 = \epsilon_1 \epsilon_2 e^3 = e^3$$
$$\star e^3 \wedge e^1 = \epsilon_3 \epsilon_1 e^2 = -e^2$$
$$\star e^2 \wedge e^3 = \epsilon_2 \epsilon_3 e^1 = -e^1$$

I think I derived these in Lecture. Note,

$$\star\star(e^1) = \star(-e^2 \wedge e^3) = -(-e^1) = e^1$$

$$\star\star(e^1 \wedge e^2) = \star(e^3) = e^1 \wedge e^2$$

$$n(n-k) = 3(3-k) = \begin{cases} 6 & k = 1 \\ 3 & k = 2 \end{cases}$$

Remark: there may be an error here.

P93 Let $SL(2) = \{ A \in \mathbb{R}^{2\times 2} / \det(A) = 1 \}$

Find coordinate chart of $SL(2)$ which includes $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2) \Rightarrow ad - bc = 1$

Suppose $d \neq 0$ then $a = \frac{1+bc}{d}$ simply use

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (b, c, d) \quad \text{where } d \neq 0 \text{ for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2)$$

This map is a bijection with inverse,

$$\psi^{-1}(b, c, d) = \begin{bmatrix} \frac{1+bc}{d} & b \\ c & d \end{bmatrix}$$

as is easily checked, for $d \neq 0$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2)$,

$$\psi\left(\psi^{-1}(b,c,d)\right) = \psi\begin{bmatrix} \frac{1+bc}{d} & b \\ c & d \end{bmatrix} = (b,c,d)$$

$$\psi^{-1}\left(\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) = \psi^{-1}(b,c,d) = \begin{bmatrix} \frac{1+bc}{d} & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as $ad - bc = 1 \Rightarrow a = \frac{1+bc}{d}$. Note $\psi\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = (0, 0, 1)$.

$d \neq 0$

p94 3.25 & 3.26 of p. 90 Rentelin

3.25 $\alpha = 3x^2y^3z\,dx + xy^3z^2\,dy + 2yz^2\,dz$

$d\alpha = 3x^2(3y^2z\,dy + y^3\,dz)\wedge dx + y^3(z^2\,dx + 2xz\,dz)\wedge dy +$
$\quad + 2z^2\,dy\wedge dz$

$d\alpha = 9x^2y^2z\,dy\wedge dx + y^3z^2\,dx\wedge dy +$
$\quad + 3x^2y^3\,dz\wedge dx + 2xy^3z\,dz\wedge dy + 2z^2\,dy\wedge dz$

$$\boxed{d\alpha = (2z^2 - 2xy^3z)\,dy\wedge dz + 3x^2y^3\,dz\wedge dx + (y^3z^2 - 9x^2y^2z)\,dx\wedge dy}$$

Let's check on $d\alpha$ by calculating $d(d\alpha) = 0$

$$d(d\alpha) = [-2y^3z + 9x^2y^2 + 2y^3z - 9x^2y^2]\,dx\wedge dy\wedge dz \neq 0$$

3.26 $w = 6x^2\,dx\wedge dy - 3xyz^3\,dy\wedge dz + x^2y^2\,dz\wedge dx$

$dw = 12x\,dx\wedge dy\wedge dy - 3(yz^3\,dx + xz^3\,dy + 3xyz^2\,dz)\wedge dy\wedge dz +$
$\quad + (2xy^2\,dx + 2x^2y\,dy)\wedge dz\wedge dx$

$$\boxed{dw = (-3yz^3 + 2x^2y)\,dx\wedge dy\wedge dz}$$

Remark: $d(dw) = 0$ is immediate from fact $\Lambda^4\mathbb{R}^3 = \{0\}$ so this does not help me check dw. In contrast, I found $d(d\alpha) = 0$ useful to hunt down a mistake I made in my first attempt (not shown).

p95 exercise 3.30 on interior product / $i_X \omega = X \lrcorner \omega$ "X hook ω"

(a.) Show $i_X\, df = df(X) = Xf$

Let $\omega = df$ where f is fnct on M $\therefore$ ω is 1-form, for vector field X,

$$i_X\, df = i_X \omega = \langle \omega, X \rangle = \omega(X) = \underbrace{df(X) = X(f)}$$

this assumes $1 = \frac{d}{dt}$ for $f: M \longrightarrow \mathbb{R}$ gives $df(X)$ a vector on $T_{f(p)}\mathbb{R}$ which we identify with $\mathbb{R}$.

(b.) verify $i_{fX} = f\, i_X$. Consider, for g a zero form

this is trivially true $i_{fX}\, g = 0 = f\, i_X\, g = f(0) = 0$.

For one-form ω,

$$i_{fX}(\omega) = \langle \omega, fX \rangle = f \langle \omega, X \rangle = f\, i_X \omega$$

Suppose, inductively, the claim for k-forms. Consider α a $k+1$ form, and vector fields $X_1, X_2, \ldots$ and f a fnct,

$$\begin{aligned} \left(i_{fX}(\alpha)\right)(X_1, \ldots, X_k) &= \alpha(fX, X_1, \ldots, X_k) \\ &= f\,\alpha(X, X_1, \ldots, X_k) \\ &= \left(f\, i_X(\alpha)\right)(X_1, \ldots, X_k) \end{aligned}$$

I didn't really use induction step here. Rather, I used the function multilinearity of tensor field (α is special case of that general family of objects, see pg. 85 ≈ Renteln)

P95

(c.) Antisymmetry of ~~inner~~ interior-product map; For ω a k-form,

$$\left(i_{X_2}\, i_{X_1}\, \omega\right)(X_3,\ldots,X_k) = \omega(X_1, X_2, \ldots, X_k) = -\omega(X_2, X_1, \ldots, X_k) = \left(-i_{X_1}\, i_{X_2}\, \omega\right)(X_3,\ldots,X_k)$$

antisymmetric multilinear map ω.

Thus, $\underline{i_{X_2}\, i_{X_1} = -\, i_{X_1 X_2}}$.

· Once more, I see no need for explicit induction, sorry.

(d.) Let $\omega = \lambda \wedge \mu \wedge \nu$ for 1-forms λ, μ, ν.
Evaluate $\omega(X_1, X_2, X_3)$.

$$\omega(X_1,X_2,X_3) = i_{X_3}\left(i_{X_2}\left(i_{X_1}\, \lambda\wedge\mu\wedge\nu\right)\right)$$
$$= i_{X_3}\left(i_{X_2}\left[(i_{X_1}\lambda)\wedge(\mu\wedge\nu) - \lambda\, i_{X_1}(\mu\wedge\nu)\right]\right)$$
$$= i_{X_3}\left(i_{X_2}\left[(i_{X_1}\lambda)\wedge\mu\wedge\nu - \lambda\wedge(i_{X_1}\mu)\wedge\nu + \lambda\wedge\mu\wedge i_{X_1}\nu\right]\right)$$
$$= \left(i_{X_3}\, i_{X_2}\, \mu\wedge\nu\right)\lambda(X_1) - \mu(X_1)\, i_{X_3}\, i_{X_2}(\lambda\wedge\nu)$$
$$+ \nu(X_1)\, i_{X_3}\, i_{X_2}(\lambda\wedge\mu)$$
$$= \lambda(X_1)\,\mu(X_2)\,\nu(X_3) - \lambda(X_1)\,\mu(X_3)\,\nu(X_2)$$
$$- \mu(X_1)\,\lambda(X_2)\,\nu(X_3) + \mu(X_1)\,\lambda(X_3)\,\nu(X_2)$$
$$+ \nu(X_1)\,\lambda(X_2)\,\mu(X_3) - \nu(X_1)\,\lambda(X_3)\,\mu(X_2)$$

aka,

$$\lambda\wedge\mu\wedge\nu = \lambda\otimes\mu\otimes\nu + \mu\otimes\nu\otimes\lambda + \nu\otimes\lambda\otimes\mu$$
$$- \nu\otimes\mu\otimes\lambda - \mu\otimes\lambda\otimes\nu - \lambda\otimes\nu\otimes\mu$$

evaluated on (X_1, X_2, X_3). $\Rightarrow$ (e.) $\boxed{C_p = 1}$

P96 Exercise 3.31 p. 95 of Renteln

Suppose $X_1, X_2, \dots, X_n$ are n-Linearly-dependent tangent vectors at $p \in M$. Show $\omega_p(X_1, \dots, X_n) = 0$ for every n-form ω. I'm ignoring the hint.

Without loss of generality we assume (if not relabel...)

$$X_1 = \sum_{j=2}^{n} c_j X_j$$

Hence, by linearity of ω_p in 1^{st} slot,

$$\omega_p(X_1, \dots, X_n) = \sum_{j=2}^{n} c_j \underbrace{\omega_p(X_j, X_2, \dots, X_n)}$$

all zero as X_j appears in the j^{th} slot for $j = 2, 3, \dots, n$ and hence X_j appears twice

$$\omega(X_j, \dots, X_j, \dots) = -\omega(X_j, \dots, X_j, \dots)$$

$$\Rightarrow \omega(X_j, \dots, X_j, \dots) = 0 \quad \forall j = 2, \dots, n.$$

P97 Ex. 3.32 and 3.33 p. 97 Renteln

$$\omega = 3xz\,dx - 7y^2z\,dy + 2x^2y\,dz$$

Consider $\phi : \mathbb{R} \longrightarrow \mathbb{R}^3$ by $\phi(t) = (t^2, 2t-1, t)$

That is $x = t^2$, $y = 2t-1$, $z = t$

$$\phi^*\omega = \underbrace{3(t^2)(t)\,d(t^2) - 7(2t-1)(t)\,d(2t-1) + 2(t^2)^2(2t-1)\,dt}_{\star}$$

$$= 6t^4\,dt - 14(2t^2 - t)\,dt + 2(2t^5 - t^4)\,dt$$

$$= \boxed{(4t^4 - 28t^2 + 14t + 4t^5)\,dt}$$

I could be more formal and sort through the def^{n},

$$(\phi^*\omega)\left(\frac{\partial}{\partial t}\right) = \omega_{\phi(t)}\left(d\phi\left(\frac{\partial}{\partial t}\right)\right) \qquad \frac{\partial}{\partial t} = \frac{d}{dt} \text{ here.}$$

$$= (3xz\,dx + 7y^2z\,dy + 2x^2y\,dz)_{\phi(t)}$$

$$\hookrightarrow \left(2t\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + 1\frac{\partial}{\partial z}\right)$$

$$= 3(t^2)(t)\,dx\left(2t\frac{\partial}{\partial x}\right) + 7(2t-1)^2 t\,dy\left(2\frac{\partial}{\partial y}\right) + 2(t^2)^2(2t-1)\,dz\left(\frac{\partial}{\partial z}\right)$$

P 97 continued

then $dx\left(2t\,\frac{\partial}{\partial x}\right) = 2t\,dx\left(\frac{\partial}{\partial x}\right) = 2t$

and $dy\left(2\,\frac{\partial}{\partial y}\right) = 2\,dy\left(\frac{\partial}{\partial y}\right) = 2$ etc and we get right back to ✡ once again.

oops! I copied wrong ϕ!

Ex. 3.33 / $\Theta = -x\,dy\wedge dz - y\,dz\wedge dx + 2z\,dx\wedge dy$

$\phi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ def$^{\underline{e}}$ by $\phi(u,v) = (u, v, u^2+v^2)$. Calculate $\phi^*\Theta$

$\mathbb{R}^2_{uv}$ has only one nontrivial two form upto scalar; $du\wedge dv$

Hence, it suffices to calculate $(\phi^*\Theta)(\partial_u, \partial_v)$

$$(\phi^*\Theta)(\partial_u,\partial_v) = \Theta_p\left(d\phi(\partial_u),\, d\phi(\partial_v)\right) \quad : \quad p = \phi(u,v).$$

$$= \Theta_p\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial x} + \frac{\partial v}{\partial u}\frac{\partial}{\partial y} + \frac{\partial(u^2+v^2)}{\partial u}\frac{\partial}{\partial z},\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial(u^2+v^2)}{\partial v}\frac{\partial}{\partial z}\right)$$

$$= \Theta_p\left(\partial_x + 2u\,\partial_z,\ \partial_y + 2v\,\partial_z\right)$$

$$= \Theta_p(\partial_x,\partial_y) + 2v\,\Theta_p(\partial_x,\partial_z) + 2u\,\Theta_p(\partial_z,\partial_y) + 4uv\,\Theta_p(\partial_z,\partial_z) \quad (=0)$$

$$= 2z + 2v(y) + 2u(x) \qquad \text{where } x = u,\ y = v,\ z = u^2+v^2$$

$$= 2(u^2+v^2) + 2v^2 + 2u^2$$

$$= 4(u^2+v^2)$$

$$\therefore\ \boxed{\phi^*\Theta = 4(u^2+v^2)\,du\wedge dv}$$

Alternatively just plug $x = u$, $y = v$ and $z = u^2+v^2$ into Θ,

$$\phi^*\Theta = -u\left(dv\wedge(2u\,du + 2v\,dv)\right) - v\left(2u\,du + 2v\,dv\right)\wedge du + 2(u^2+v^2)\,du\wedge dv$$

$$= 2u^2\,du\wedge dv + 2v^2\,du\wedge dv + 2(u^2+v^2)\,du\wedge dv$$

$$= \underline{4(u^2+v^2)\,du\wedge dv}$$

I think it is easy enough to see which approach is most computationally efficient. I include both to help convince you they're the same.

P97 again 3.33 with actual ϕ

$$\phi(u,v) = (\underbrace{u^3 \cos v}_{x}, \underbrace{u^3 \sin v}_{y}, \underbrace{u^2}_{z}),$$

$$\theta = -x\, dy \wedge dz - y\, dz \wedge dx + 2z\, dx \wedge dy$$

$$\phi^*\theta = -x(3u^2 \sin v\, du + u^3 \cos v\, dv) \wedge (2u\, du)$$
$$-y(2u\, du) \wedge (-u^3 \sin v\, dv)$$
$$-2z(3u^2 \sin v\, du + u^3 \cos v\, dv) \wedge (3u^2 \cos v\, du - u^3 \sin v\, dv)$$

$$= \left[2xu^4 \cos v + 2yu^4 \sin v - 2z(-3u^5 \sin^2 v - 3u^5 \cos^2 v)\right] du \wedge dv$$

$$= (2u^7 \cos^2 v + 2u^7 \sin^2 v + 6u^7)\, du \wedge dv$$

$$= \boxed{8u^7\, du \wedge dv}$$

$$= d(u^8 dv) \qquad \text{hmm.}$$

P98 Let $w = 4y\,dx + 2xy\,dy + dz$ and define $\phi: \mathbb{R}^2 \to \mathbb{R}^3$ by $\phi(u,v) = (u, v, u^2+v^2)$. Show that $d(\phi^* w) = \phi^*(dw)$

Ex. 3.34 pg. 97 Renteln

$$\phi^* w = 4(v)\,du + 2uv\,dv + d(u^2+v^2) = (4v+2u)du + (2uv+2v)dv$$

$$d(\phi^* w) = 4\,dv\wedge du + 2v\,du\wedge dv = \underline{(2v-4)\,du\wedge dv} \quad \text{(I.)}$$

In contrast,

$$dw = 4\,dy\wedge dx + 2y\,dx\wedge dy = (2y-4)\,dx\wedge dy$$

Hence,

$$\phi^*(dw) = \underline{(2v-4)\,du\wedge dv} \quad \text{II.}$$

as (I.) matches (II.) we derive $\phi^*(dw) = d(\phi^* w)$ as desired.

P99 Ex. 3.40 p. 104 of Renteln

- Beware θ is ϕ and ϕ is θ in Renteln

S^2 has area element $\sigma = \sin\theta\, d\theta \wedge d\phi$

Let $f: \mathbb{R}^3 - \{0\} \longrightarrow S^2$ be def^n by $f(\vec{x}) = \frac{\vec{x}}{\|x\|} = \frac{\vec{x}}{r}$

where $r = \sqrt{x^2+y^2+z^2}$. Oh, $f(x,y,z) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$

(a.) $f^*\sigma = ?$ $\quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} - \frac{x}{r^2}\frac{x}{r}$, $\frac{\partial}{\partial x}\left(\frac{y}{r}\right) = \frac{-xy}{r^3}$, $\frac{\partial}{\partial x}\left(\frac{z}{r}\right) = \frac{-xz}{r^3}$

$$(f^*\sigma)_q(\partial_x, \partial_y) = \sigma_p\left(df(\partial_x), df(\partial_y)\right) \quad : p = f(q)$$

$$= \sigma_p\left(\left(\frac{1}{r} - \frac{x^2}{r^3}\right)\partial_x - \frac{xy}{r^3}\partial_y - \frac{xz}{r^3}\partial_z,\ \frac{-yx}{r^3}\partial_x + \left(\frac{1}{r} - \frac{y^2}{r^3}\right)\partial_y - \frac{yz}{r^3}\partial_z\right)$$

I think I'll try to do this w/o cartesians.

P99 continued

$$f(r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$$

$$\sigma = \sin\theta\, d\theta\wedge d\phi$$

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

Since $r=1$ on S^2 have $dr=0$

$$dx\wedge dy = r^2(-\sin\phi\sin\theta\, d\phi + \cos\phi\cos\theta\, d\theta)\wedge(\cos\phi\, d\phi\sin\theta + \sin\phi\cos\theta\, d\theta)$$
$$= r^2(\cos^2\phi\sin\theta\cos\theta + \sin^2\phi\sin\theta\cos\theta)\, d\theta\wedge d\phi$$
$$= r^2\sin\theta\cos\theta\, d\theta\wedge d\phi$$
$$= \underline{(r\sin\theta)\, z\, d\theta\wedge d\phi} \quad (I)$$

$$dy\wedge dz = r^2(\cos\phi\sin\theta\, d\phi)\wedge(-\sin\theta\, d\theta) = \underline{(r\sin\theta)\, x\, d\theta\wedge d\phi} \quad (II)$$

$$dz\wedge dx = r^2(-\sin\theta\, d\theta)\wedge(-\sin\phi\sin\theta\, d\phi) = \underline{(r\sin\theta)\, y\, d\theta\wedge d\phi} \quad (III)$$

Hence,

$$\sin\theta\, d\theta\wedge d\phi = \frac{1}{rz}\, dx\wedge dy\,,\ \frac{1}{rx}\, dy\wedge dz\,,\ \frac{1}{ry}\, dz\wedge dx$$

Remark: unfinished

P100 3.42 on p. 105 of Renteln

$f: M \longrightarrow N$ smooth map of manifolds
$m = \dim M \geq \dim N = n$ and $q \in N$ a regular value of f
Let $\Sigma = f^{-1}(q)$ and suppose $p \in \Sigma$. Show that
$T_p \Sigma$ is the kernel of the differential $f_* : T_p M \longrightarrow T_q N$

To say $q \in N$ is a regular value means f is a submersion at each $p \in f^{-1}(q)$. This means the Jacobian has maximal rank (namely n) at each $p \in f^{-1}(q)$. However,

$$f_* : T_p M \longrightarrow T_q N \quad \text{as} \quad f(p) = q \quad \text{for } p \in f^{-1}(q)$$

and $\underbrace{\operatorname{rank}(f_*)}_{n} + \dim(\ker(f_*)) = \underbrace{\dim(T_p M)}_{m}$

$$\therefore \quad \dim(\ker f_*) = m - n.$$

However, $\Sigma = f^{-1}(q)$ is smooth submanifold of M with $\dim(\Sigma) = m - n$ by Th^{m} (3.2) on pg. 72 of Renteln.

Furthermore, if $v \in T_p \Sigma$ then calculate,

$$f_*(v) = v[f] = 0$$

as f is constant $f(p) = q \quad \forall p \in \Sigma$. Thus

$T_p \Sigma \subset \ker(f_*)$. But, as $\dim(T_p \Sigma) = \dim(\ker f_*) = m - n$

it follows $T_p \Sigma = \ker f_*$.

P101 Ex. 3.4.3 p. 105-106 Renteln

(a.) Prove $Gl(n,\mathbb{R}) = \{A \in \mathbb{R}^{n^2 \times n} \mid \det(A) \neq 0\}$ is an n^2-dim'l Lie group

We have $\mathbb{R}^{n^2 \times n}$ is an n^2-dim'l manifold with global chart $X_{ij}(A) = A_{ij}$. Moreover, matrix addition & multiplication is smooth w.r.t. X_{ij}-coordinates. Naturally $Gl(n,\mathbb{R}) \subset \mathbb{R}^{n \times n}$ and we argue that matrix multiplication and inversion are smooth since the formulas for $(A,B) \longmapsto AB$ and $A \longmapsto A^{-1}$ are rational functions of the coordinates $X_{ij}(A)$, $X_{ij}(B)$. In particular, think of $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)^T$. To see $Gl(n,\mathbb{R})$ is a manifold we argue

$$Gl(n,\mathbb{R}) = \underbrace{\det^{-1}\{(-\infty,0) \cup (0,\infty)\}}_{\text{inverse image of open set is open under continuous map } \det}$$

Thus $Gl(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n \times n}$ for which matrix multiplication & inversion restricted to $Gl(n,\mathbb{R})$ are smooth. $\therefore$ $Gl(n,\mathbb{R})$ is Lie Group.

(b.) Show that $O(n) = \{A \in \mathbb{R}^{n \times n} \mid A^T A = I\}$ is a Lie group.

Consider $\varphi(A) = AA^T$ notice $\varphi : M_n(\mathbb{R}) \longrightarrow \underbrace{M_n^+(\mathbb{R})}_{\text{Symmetric}}$

We wish to argue $q = I$ is regular value of φ then Thm 3.2 says $\varphi^{-1}(I) = O(n) = \{A \mid A^T A = I\}$ is a submanifold of $M_n(\mathbb{R}) = \mathbb{R}^{n \times n}$

P101 continued

$$M_n^+(\mathbb{R}) = \{ A \mid A^T = A \} = \text{span} \underbrace{\{ E_{ij} + E_{ji} \mid i < j \} \cup \{ E_{ii} \}_{i=1}^n}_{\frac{1}{2}(n^2+n) \text{ - elements.}}$$

We wish to show $q = I$ is regular value for $\varphi(A) = AA^T$

This requires J_φ have rank $\frac{1}{2}(n^2+n)$

$$\begin{aligned} \varphi(A+H) &= (A+H)(A+H)^T \\ &= (A+H)(A^T+H^T) \\ &= AA^T + HA^T + AH^T + HH^T \end{aligned}$$

Identifying $T_p M_n(\mathbb{R}) = M_n(\mathbb{R})$ we argue,

$$d_A\varphi(H) = HA^T + AH^T$$

It is clear from the usual arguments given that $\varphi(A+H) - \varphi(A) = HA^T + AH^T + HH^T$. Notice

$$\text{Ker}(d_A\varphi) = \{ H \mid HA^T + AH^T = 0 \}$$

In the case $A = I$,

$$\text{Ker}(d_I\varphi) = \underbrace{\{ H \mid H + H^T = 0 \}}_{\text{antisymmetric matrices (complementary to } M_n^+(\mathbb{R}))} = M_n^-(\mathbb{R})$$

$$M_n^+(\mathbb{R}) \oplus M_n^-(\mathbb{R}) = M_n(\mathbb{R})$$

Thus, $\dim(\text{Ker } d_I\varphi) = \frac{1}{2}(n^2-n)$ from which we deduce $\dim(\text{im}(d_I\varphi)) = \frac{1}{2}(n^2+n) \Rightarrow q = I$ is regular pt.

Thus Th^m 3.2 applies and $\varphi^{-1}(I) = O(n)$ is submanifold of $M_n(\mathbb{R})$ of dimension $n^2 - \frac{1}{2}(n^2+n) = \boxed{\frac{1}{2}(n^2-n) = \dim(O(n))}$

P101 continued

(c.) $SO(n) = \{A \in O(n) \mid \det A = 1\}$ = special orthogonal group

Consider $g : O(n) \longrightarrow \mathbb{R}$ defined by $g(A) = \det(A)$

Notice $A \in O(n) \Rightarrow A^TA = I \Rightarrow \det A^T \det A = (\det A)^2 = \det(I) = 1$

Hence, $\det A = \pm 1$. In particular, $g^{-1}\{1\} = SO(n)$

and we may use Thm 3.2 as it is easy to see $g = 1$ is a regular value (see Ex. 1.36 on p 23 where we learned $\frac{\partial}{\partial A_{ij}}(\det A) = \det(A)(A^{-1})_{ji}$)

$$d_A g(H) = \sum_{i,j} \det(A)(A^{-1})_{ji} H_{ij} \neq 0 \quad \text{for } H \neq 0$$

well, in any event $d_A g$ has rank 1 so Thm 3.2 applies.

In fact, $g^{-1}\{-1\} = -SO(n)$ is also a submanifold of $O(n)$ of codimension 1, but only $SO(n)$ is a Lie subgroup as $SO(n)$ contains I.

(d.) Show $SO(2)$ is diffeomorphic to circle S^1 (what follows is not 100% complete)

If $A \in SO(2)$ then $A^TA = I$ and $\det A = 1$. I invite the reader to prove there exists $\theta \in \mathbb{R}$ for which $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Define $\varphi\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = (\cos\theta, \sin\theta)$

Alternatively, $\varphi\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a, b)$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2)$

$\varphi^{-1}(a,b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ has $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}^T \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$

provided $a^2+b^2 = 1$ we have $\varphi^{-1}(a,b) \in SO(2)$ as $\det\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a^2+b^2 = 1$.

We have φ is a bijection between $SO(2)$ and S^1 which is smooth thus $SO(2)$ is diffeomorphic to S^1.

P102 Ex. 3.44 p. 106-107

Let G be Lie group with identity $e \in G$.
Define $L_g : G \longrightarrow G$ by $L_g(h) = gh$ (left multiplication by g)
A vector field X is <u>left-invariant</u> if $(L_g)_* (X) = X \quad \forall g \in G$
meaning, $(L_g)_* X_h = X_{gh} \quad \forall g, h \in G$.

(a.) if X, Y are left-invariant then consider $[X, Y]$ by Th^{m} 3.7

$$\begin{aligned}(L_g)_* ([X, Y]_h) &= [\, (L_g)_* (X_h), (L_g)_* (Y_h)\,] \\ &= [\, X_{gh}, Y_{gh}\,] \\ &= [X, Y]_{gh}\end{aligned}$$

(b.) $(L_g)_* X_e = X_{ge} = X_g$ thus X_g is pushed-forward by $(L_g)_*$ from the value X_e

Let $Y \in T_e G$ then define $X_g = (L_g)_* (Y) \quad \forall g \in G$.

Thus $T_e G$ obtains a Lie Algebra with bracket induced from the Lie bracket of left-invariant vector fields; $[X_e, Y_e] \overset{\text{def}^n}{=} [X, Y]_e$

(c.) Show $\exists$ a bijection between curves $\gamma : \mathbb{R} \longrightarrow G$ with $\gamma(s+t) = \gamma(s)\gamma(t)$ and the Lie algebra $\mathfrak{g} = T_e G$. Consider the integral curve γ of a left-invariant vector field X, we have

$$X(\gamma(t)) = \frac{d\gamma}{dt}$$

- Well, sorry folks, this part (c.) is a bit more than I intended. If you're interested I have a sol^{n} somewhere But, I'm going on for now.

P102 new 3.46 on p. 108 / I'm changing it for those who haven't done this already.

102 revised edition

Problem 3.46 p. 108 Renteln

Show that $\mathfrak{so}(n)$, the Lie algebra of $SO(n)$ can be identified with $(M_n^-(\mathbb{R}), [,])$ where $M_n^-(\mathbb{R}) = \{B \mid B^T = -B\} \subset \mathbb{R}^{n\times n}$

Let $\gamma : \mathbb{R} \longrightarrow SO(n)$ be a curve with $\gamma(0) = I$

then $\gamma(t)^T \gamma(t) = I$ for $t \in \mathbb{R}$. Differentiate,

$$\frac{d\gamma}{dt}^T \gamma(t) + \gamma(t)^T \frac{d\gamma}{dt} = 0$$

Thus, for $\gamma'(0) \in T_I SO(n) = \mathfrak{so}(n)$ we have

$$\gamma'(0)^T I + I^T \gamma'(0) = 0 \implies \underline{\gamma'(0)^T = -\gamma(0)}. \;//$$

Problem 3.51 p. 109 Renteln

If $A, B \in M_n(\mathbb{R})$ and $[A,[A,B]] = [B,[A,B]] = 0$ then $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$. So, $e^A e^B = e^{A+B}$ when $[A,B] = 0$.

Consider $e^{t(A+B)} = e^{tA} f(t)$ and show $f(t) = e^{tB} e^{\frac{-t^2}{2}[A,B]}$

$$(A+B)e^{t(A+B)} = Ae^{tA} f(t) + e^{tA} \frac{df}{dt}$$

But,

$$Ae^{tA} e^{tB} e^{\frac{-t^2}{2}[A,B]} + e^{tA}\left(Be^{tB} e^{\frac{-t^2}{2}[A,B]} + e^{tB}(-t[A,B]) e^{\frac{-t^2}{2}[A,B]} \right)$$

P106 Let $F(x,y,z) = (\cos(x), \sin(xy))$.

Calculate the push-forward of $X = a\partial_x + b\partial_y + c\partial_z$ under F at (x,y,z)

Let $F : \mathbb{R}^3_{xyz} \longrightarrow \mathbb{R}^2_{uv}$ as defined above,

$$dF(X) = X(u\circ F)\frac{\partial}{\partial u} + X(v\circ F)\frac{\partial}{\partial v}$$

$$= (a\partial_x + b\partial_y + c\partial_z)(\cos x)\frac{\partial}{\partial u} + (a\partial_x + b\partial_y + c\partial_z)(\sin xy)\frac{\partial}{\partial v}$$

$$= \boxed{-a\sin(x)\frac{\partial}{\partial u} + (ay\cos(xy) + bx\cos(xy))\frac{\partial}{\partial v}}$$

P107 Write $\omega_{\vec{F}}$ and $\Phi_{\vec{F}}$ in terms of the musical morphism and Hodge dual $\star$ with respect to Euclidean metric & $dx\wedge dy\wedge dz$

$$\star dx = dy\wedge dz \qquad \star dx\wedge dy = dz$$
$$\star dy = dz\wedge dx \qquad \star dz\wedge dx = dy$$
$$\star dz = dx\wedge dy \qquad \star dx\wedge dy = dz$$

$$\longrightarrow \quad \star\omega_{\vec{F}} = \Phi_{\vec{F}}, \quad \star\Phi_{\vec{F}} = \omega_{\vec{F}}$$

In this special case, $\star\star\alpha = \alpha$.

$$\omega_{\langle a,b,c\rangle} = a\,dx + b\,dy + c\,dz = b\langle a,b,c\rangle$$

or, if you wish, $b(a\partial_x + b\partial_y + c\partial_z) = a\,dx + b\,dy + c\,dz$

Anyway, whichever notation you use, $\boxed{\omega_{\vec{F}} = b\vec{F}.}$

$$\Phi_{\vec{F}} = a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy$$
$$= \star(\omega_{\vec{F}}) \quad \text{for } \vec{F} = \langle a,b,c\rangle$$
$$= \star(b\vec{F}) \quad \therefore \quad \boxed{\Phi_{\vec{F}} = \star(b\vec{F})}$$

P108 derive formulas for gradient in spherical coordinates.

$$\omega_{\nabla f} = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \frac{\partial f}{\partial \rho}d\rho + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial \theta}d\theta$$

As we discussed, $(g^{ij}) = \begin{bmatrix} 1 & & \\ & \rho & \\ & & \rho\sin\phi \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ & 1/\rho & \\ & & 1/\rho\sin\phi \end{bmatrix}$ ↻

$$\#\, d\rho = \partial/\partial\rho, \quad \#(\rho\, d\phi) = \partial/\partial\phi, \quad \#(\rho\sin\phi\, d\theta) = \partial/\partial\theta$$

Thus,

$$\omega_{\nabla f} = \#\, b\left(\frac{\partial x}{\partial \rho}\frac{\partial}{\partial x} + \frac{1}{\rho}\frac{\partial \phi}{\partial \rho}\frac{\partial}{\partial \phi} + \frac{1}{\rho\sin\phi}\frac{\partial \theta}{\partial \rho}\frac{\partial}{\partial \theta}\right)$$

Which is to say,

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\hat{\theta}$$