Same instructions as Mission 1. Thanks! *What follows is based on an amalgam of my 2015 Lecture Notes Chapter 9-10, Paul Renteln and Jeffrey Lee's texts.*

Problem 130 Your signature below indicates you have:

(a.) I read most of of Cook's Chapter 10 and 11: ________________.

(b.) I read Renteln §8.9: ________________.

Problem 131 We define the coderivative $\delta : \Lambda^p M \to \Lambda^{p-1} M$ by:

$$\delta\alpha = (-1)^{np+n+1} \star d \star \alpha$$

Observe $\delta\alpha$ is a differential form of one-less degree than α. In contrast, $d\alpha$ is a differential form of one-more degree than α. We proved, or observed, that $d^2 = 0$ and $\star\star = \pm 1$ where the sign depends both on the signature of the metric on M as well as the degree of the form and the dimension of M. However, those details need not concern us for the following question: **Show:**

$$\delta^2\alpha = 0$$

Problem 132 In the previous problem I was intentionally vague about which manifold and metric we were using. Here, I wil be precise: let us consider $M = \mathbb{R}^3$ with the Euclidean metric. Find nice formulas for: here $\omega_{\vec{F}} = Pdx + Qdy + Rdz$ as usual,

(a.) $d\delta\omega_{\vec{F}}$ where $\vec{F} = \langle P, Q, R\rangle$.

(b.) $\delta d\omega_{\vec{F}}$ where $\vec{F} = \langle P, Q, R\rangle$.

(c.) is $d\delta + \delta d$ anything interesting?

Problem 133 Suppose α is a p-form on $\mathbb{R}^3$. Let $\star$ denote the standard euclidean Hodge dual. Is it possible that $\star d(\alpha) = d(\star\alpha)$?

Problem 134 The Hodge dual operation on $\mathbb{R}^3$ allows us to intoduce another way to differentiate a p-form α:

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha = (-1)^{3p} * d * \alpha$$

where I have set $n = 3$ since I intend to use this **coderivative** δ on $\mathbb{R}^3$. Let $\alpha = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$ where a, b, c are smooth functions on $\mathbb{R}^3$. Calculate the formula for $\delta\alpha$.

Problem 135 De Rahm, Hodge and others developed a theory to analyze closed vs. exact differential forms. See my notes for an example of how the shape of the domain can come into play. One interesting theorem Hodge proved was that if ω was any p-form on a Riemannian manifold then there exists a $(p-1)$-form α and a $(p+1)$-form β and a *harmonic form* γ such that

$$\omega = d\alpha + \delta\beta + \gamma.$$

In the special case $M = \mathbb{R}^3$ it is the case $\gamma = 0$. **Use the theorem due to Hodge to prove that any vector field can be written in terms of the gradient of a scalar**

function and the curl of some vector field; that is, for any vector field $\vec{F}$ there exists another vector field $\vec{G}$ and a function g such that $\vec{F} = \nabla g + \nabla \times \vec{G}$. I think if you examine the case $\omega = \omega_{\vec{F}}$ then it ought to be about a line or two once you unravel the notation. I let Hodge do the really hard part. (you need to use the preceding problem to understand the coderivative part)

Problem 136 Consider $\omega = (x+y)dx + (y+z)dy + (z+x)dz$ on $\mathbb{R}^3$. Verify Hodge's Theorem (see preceding problem) by finding α and β such that $\omega = d\alpha + \delta\beta$. Begin your quest by understanding what the degrees of α and β must be in your context.

Warning: the notations *, $\star$ and $_*$ mean different things. I do not use $_*$, but other authors use $df(v) = f_*(v)$; that is f_* is the push-forward by f. I use f^* as the pull-back by f and finally $\star\gamma$ is the Hodge dual of γ which is only defined with respect to a metric (and for us, either $\mathbb{R}^3$ euclidean or $\mathbb{R}^4$ Minkowski for the application to Electrodynamics in a later chapter). Hodge duality is far less basic than the other two stars of this discussion. If you wish to read on Hodge duality in some generality then you might look at David Bleecker's text *Gauge Theory and Variational Principles* which is inexpensive in Dover format.

Problem 137 The rank of a one-form ω is defined, at a point, to be the largest positive integer r for which $\omega_r \neq 0$ yet $\omega_{r+1} = 0$ where we define the **auxillary forms** $\omega_1, \omega_2, \dots$ as follows:

$$\omega_1 = \omega, \ \omega_2 = d\omega, \ \omega_3 = \omega \wedge d\omega, \ \omega_4 = d\omega \wedge d\omega, \ \omega_5 = \omega \wedge d\omega \wedge d\omega, \dots$$

The auxillary forms $\{\omega_1, \dots, \omega_r\}$ form a LI set of forms in the exterior algebra of the manifold at a the given point. If the rank of the one-form is constant at all points then the form is called **regular**.Find the rank of:

$$\omega = (x+y)dt + ydz$$

on $\mathbb{R}^4_{txyz}$ space.

Problem 138 Consider the one-form $\omega = xdx + ydy + zdz$ on $\mathbb{R}^3$. Find the foliation of three dimensional space into two-dimensional submanifolds whose tangent spaces are spanned by vector fields which are found in $\ker(\omega)$. Check the condition needed to show ω is dual to a two-plane field distribution on $\mathbb{R}^3$; that is verify $\omega \wedge d\omega = 0$ (see Bachman for where I'm coming from here). Incidentally, there is one point left out, perhaps it would be more honest to say find a foliation of $\mathbb{R}^3 - \{(0,0,0)\}$.

Problem 139 Consider the linear differential equation $\frac{dy}{dx} + Py = Q$ where P, Q are smooth functions of x. Or, $\omega = 0$ where the Pfaffian form:

$$\omega = dy + (Py - Q)dx.$$

What condition is needed for μ if $\mu\omega$ is to be an exact form? Does your result correspond to what you learned in calculus II about the integrating factor method? (ask me about the integrating factor method if you didn't see it before) Hint: $d^2 = 0$.

Problem 140 Consider $\omega = dy + dz + xydx + xzdx$. Show that $\omega \wedge d\omega = 0$ on all of $\mathbb{R}^3$. What foliation of $\mathbb{R}^3$ does ω describe. Recall, we discussed that $\omega = dz$ corresponds to foliating $\mathbb{R}^3$ into $z = c$ (a family of horizontal planes, each leaf in the foliation labeled by c). Try to find the corresponding family of surfaces for the ω given here.

Problem 141 (Lee p.389 Problem 11) Let ω below defined for $(x, y, z) \neq 0$,

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

Calculate $d\omega$. Is ω closed? Is ω exact? Express ω in spherical coordinates (pull-back dx, dy, dz etc. to $d\rho, d\phi, d\theta$ etc.)

Problem 142 (Lee p.389 Exercise 9.36) Let $X : U \to V \subseteq S$ parametrize $S \subseteq \mathbb{R}^3$ and define $N = \partial_u X \times \partial_v X$ where (u, v) are coordinates in U (they're parameters of S). Let $dS = i_N \mathrm{vol}|_S$ where $\mathrm{vol} = dx \wedge dy \wedge dz$. **Show** $X^* dS = ||\partial_u X \times \partial_v X|| du \wedge dv$.

Problem 143 **Prove:** A nowhere vanishing smooth vector field exists on the sphere S^n iff n is odd. (hints to follow)

Problem 144 Exercise 4.11 on page 134-136 of Renteln (developing the fundamental group)

Problem 145 Exercise 5.9 on page 154-155 of Renteln (surgery on donuts)

Problem 146 Let $G(x, y, z, w) = (x^2 + y^2,\ zw,\ y + z)$. Define $M = G^{-1}\{(25, 1, 0)\}$. Find the tangent and normal spaces to M at the point $(3, 4, -4, -1/4)$.

Problem 147 We define $SU(n) = \{A \in \mathbb{C}^{n\times n} \mid det(A) = 1,\ A^\dagger A = I\}$.

(a) show $SU(n)$ is a group.

(b) show $SU(2)$ is a manifold by providing an atlas.

The following problems will only make sense if I get a chance to discuss how to algebratize differential equations. I hope I find time as it is an interesting application of symmetric tensor product. I hope to post a supplement based on Cartan for Beginners which makes these accessible (these are pretty easy once you have the appropriate back story)

Problem 148 Calculate the dimension of $J^2(\mathbb{R}^3, \mathbb{R})$

Problem 149 Calculate the dimension of $J^3(\mathbb{R}^2, \mathbb{R}^2)$

Problem 150 Express $u_{xx} + u_{yy} = 0$ as a Tablau.

Problem 151 Prolong the Cauchy Riemann equation Tablau as to obtain the Laplace equation.