Show your work and justify steps.

Problem 1 [5pts] Let $v, w \in \mathbb{R}^n$. Suppose $v \bullet w = 0$. Show that $||v + w||^2 = ||v||^2 + ||w||^2$.

Problem 2 [5pts] Give an example of a function which is differentiable, but not continuously differentiable at a point.

Problem 3 [5pts] Define what it means for $U \subseteq V$ to be an open set in V where V is a normed linear space with norm $|| \cdot ||$.

Problem 4 [10pts] Suppose $F(x, y) = (xy, x^2y^2, x^3y^3)$ and $G(a, b, c) = (a + b, \sqrt{b + c})$. If $H = G \circ F$ then calculate $H'(x, y)$. You may leave your answer in terms of the product of two appropriate matrices. However, be sure the entries in the matrices are correct.

Problem 5 [5pts] Find the standard matrix of $T(x, y) = (x + 2y, 3x + 4y)$.

Problem 6 [10pts] Suppose $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$. Calculate $\left(\frac{\partial w}{\partial x}\right)_y$.

Problem 7 [10pts] If $a \in (0, \infty)$ then it can be shown that

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}.$$

Use the fact given above to derive a nice formula for

$$\int_{-\infty}^{\infty} x^4 e^{-ax^2}\, dx.$$

Problem 8 [10pts] Let $F(x, y, z) = (x^2 + y^2 + z^2, x^2)$. Find a parametrization(s) of the curve(s) $F^{-1}\{(4, 1)\}$.

Problem 9 [10pts] Suppose $G(x, y, z) = (xy, yz)$. Answer the following questions without solving any equations. Instead, use a theorem to justify your claims:

(a.) where is it possible to locally solve $G(x, y, z) = (1, 1)$ for y, z as functions of x

(b.) where is it possible to locally solve $G(x, y, z) = (1, 1)$ for x, z as functions of y

Problem 10 [10pts] Let $F(x, y, z) = (x+y, x^2+y^2, x^3+y^3+z^3)$. Calculate $F'(x, y, z)$ and find where F is locally invertible.

Problem 11 [10pts] Find extrema of $f(x, y) = 2x^2 + 4y^2$ on the unit-circle $x^2 + y^2 = 1$.

Problem 12 [10pts] Let $\Phi(t) = (t, t^2, t^3, t^4)$ for all $t \in \mathbb{R}$. Define $C = \Phi(\mathbb{R})$. Let $p = (1, 1, 1, 1)$. Derive the tangent and normal spaces to C at p; that is calculate T_pC and N_pC. You may describe T_pC and N_pC as a span or as point-sets in $\mathbb{R}^4$ given by cartesian equations, your choice.

Problem 13 [10pts] Suppose functions of the form $f : \mathbb{R}^3 \to \mathbb{R}$ have multivariate power series expansions centered at p as given below. In each case, identify $\nabla f(p)$ and decide if p is a critical point. If p is a critical point then use the theory of quadratic forms to classify the extrema.

(a.) $p = (1, 2, 3)$,

$$f(x, y, z) = 10+(x-1)+2(z-3)+(x-1)^2+(y-2)^2+\cdots$$

(b.) $p = (1, 2, 4)$,

$$f(x, y, z) = 3-(x-1)^2-(y-2)^2-4(z-4)^2+\cdots$$

(c.) $p = (0, 0, 0)$, hint, try $\lambda = -1$.

$$f(x, y, z) = 3+2xy+2xz+2yz\cdots$$

Problem 14 [10pts] Let V, W be finite dimensional vector spaces with norms $||\cdot||_V$ and $||\cdot||_W$ respectively. Assume F, G are differentiable functions from V to W which are (Frechet) differentiable at p. Show that $F+G$ is also differentiable at p.

Problem 15 [10pts] Let $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $G(x, y) = x^T J y$ for some invertible matrix J. Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has $T(x) = Ax$ for all $x \in \mathbb{R}^n$. If $G(T(x), T(y)) = G(x, y)$ for all $x, y \in \mathbb{R}^n$ then what condition does this force A to satisfy?

Problem 16 [10pts] Calculate $dG_{(v,w)}(H, K)$ for G given in the previous problem.

Show your work and justify steps.

Problem 17 [10pts] Let V, W be finite dimensional vector spaces with norms $||\cdot||_V$ and $||\cdot||_W$ respectively. Suppose $T : V \to W$ is a linear transformation. Show T is continuous.

Problem 18 [10pts] Suppose $R \in \mathbb{R}$ is a fixed, positive constant. Let $X : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by

$$X(\theta, \phi, \psi) = (R\cos\theta\sin\phi\sin\psi, R\sin\theta\sin\phi\sin\psi, R\cos\phi\sin\psi, R\cos\psi).$$

Let $X(\mathbb{R}^3) = V$. Let $F(x, y, z, t) = x^2 + y^2 + z^2 + t^2$. Show that $V = F^{-1}\{R\}$. Let $p = X(\pi/4, \pi/4, \pi/6)$. Find T_pV and N_pV. You may describe T_pC and N_pC as a span or as point-sets in $\mathbb{R}^4$ given by cartesian equations, your choice.

Problem 19 [10pts] Suppose we wish to find the extrema of $F : \mathbb{R}^n \to \mathbb{R}$ on some compact domain given by $G^{-1}\{0\}$ where $G = (G_1, \ldots, G_p) : \mathbb{R}^n \to \mathbb{R}^p$. Consider the function $H(x, \lambda_1, \ldots, \lambda_p) = F(x) - \sum_{i=1}^{p} \lambda_i G_i(x)$ where $H : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$. Explain what critical points of H yield.