Show your work and justify steps.

Problem 1 [5pts] Let $v, w \in \mathbb{R}^n$. Suppose $v \bullet w = 0$. Show that $||v+w||^2 = ||v||^2 + ||w||^2$.

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$$\begin{aligned} ||v+w||^2 &= (v+w)\cdot(v+w) \\ &= v\cdot v + \underbrace{\cancel{v\cdot w} + \cancel{w\cdot v}}_{0} + w\cdot w \\ &= ||v||^2 + ||w||^2. \end{aligned}$$

Problem 2 [5pts] Give an example of a function which is differentiable, but not continuously differentiable at a point.

$$\underbrace{f(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}}_{\text{Standard Example.}} \qquad f'(x) = \begin{cases} 0 \\ 2x\sin\left(\frac{1}{x}\right) - \underbrace{\cos\left(\frac{1}{x}\right)}_{\text{not continuous!}} \end{cases}$$

Problem 3 [5pts] Define what it means for $U \subseteq V$ to be an open set in V where V is a normed linear space with norm $||\cdot||$.

If each point in U is interior then U is open. This means $\exists \delta > 0$ s.t. $B_\delta(u) = \{x \in V \mid ||x-u|| < \delta\}$ has $B_\delta(u) \subseteq U$ for each $u \in U$.

Problem 4 [10pts] Suppose $F(x,y) = (xy, x^2y^2, x^3y^3)$ and $G(a,b,c) = (a+b, \sqrt{b+c})$. If $H = G \circ F$ then calculate $H'(x,y)$. You may leave your answer in terms of the product of two appropriate matrices. However, be sure the entries in the matrices are correct.

$$F'(x,y) = \begin{bmatrix} y & x \\ 2xy^2 & 2yx^2 \\ 3x^2y^3 & 3y^2x^3 \end{bmatrix} \qquad G'(a,b,c) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{2\sqrt{b+c}} & \frac{1}{2\sqrt{b+c}} \end{bmatrix}$$

$$\begin{aligned} H'(x,y) &= (G\circ F)'(x,y) \\ &= G'(F(x,y))\,F'(x,y) \\ &= G'(xy, x^2y^2, x^3y^3)\,F'(x,y) \end{aligned}$$

$$\Rightarrow H'(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2\sqrt{x^2y^2+x^3y^3}} & \frac{1}{2\sqrt{x^2y^2+x^3y^3}} \end{bmatrix} \begin{bmatrix} y & x \\ 2xy^2 & 2yx^2 \\ 3x^2y^3 & 3y^2x^3 \end{bmatrix}$$

Problem 5 [5pts] Find the standard matrix of $T(x,y) = (x+2y, 3x+4y)$.

$$[T] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

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Problem 6 [10pts] Suppose $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$. Calculate $\left(\frac{\partial w}{\partial x}\right)_y$.

$dw = 2x\,dx + 2y\,dy + 2z\,dz \qquad dz = 2x\,dx + 2y\,dy$

want to isolate dw with no dz.

$$dw = 2x\,dx + 2y\,dy + 2z(2x\,dx + 2y\,dy)$$
$$= \underbrace{[2x + 4zx]}\,dx + [2y + 4zy]\,dy$$
$$\hookrightarrow \boxed{\left(\frac{\partial w}{\partial x}\right)_y = 2x + 4zx}$$

Problem 7 [10pts] If $a \in (0, \infty)$ then it can be shown that

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}.$$

Use the fact given above to derive a nice formula for

$$I = \int_{-\infty}^{\infty} x^4 e^{-ax^2}\,dx.$$

$$\frac{\partial}{\partial a}\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial a}\left(e^{-ax^2}\right)dx = \int_{-\infty}^{\infty} -x^2 e^{-ax^2}\,dx$$

$$\Rightarrow \frac{\partial^2}{\partial a^2}\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \int_{-\infty}^{\infty} x^4 e^{-ax^2}\,dx.$$

However, $\frac{\partial^2}{\partial a^2}\left(\sqrt{\frac{\pi}{a}}\right) = \sqrt{\pi}\,\frac{\partial^2}{\partial a^2}\left(a^{-\frac{1}{2}}\right) = \sqrt{\pi}\left(\frac{-1}{2}\right)\left(\frac{-1}{3}\right)a^{-5/2} \quad \therefore \boxed{I = \frac{\sqrt{\pi}}{6a^{5/2}}}$

Problem 8 [10pts] Let $F(x, y, z) = (x^2 + y^2 + z^2, x^2)$. Find a parametrization(s) of the curve(s) $F^{-1}\{(4, 1)\}$.

$$(x,y,z) \in F^{-1}\{(4,1)\} \Rightarrow \begin{cases} x^2+y^2+z^2=4 \\ x^2=1 \end{cases} \Rightarrow x = \pm 1 \Rightarrow \underline{y^2+z^2=3}$$

$y = \sqrt{3}\cos t, \; z = \sqrt{3}\sin t$

$$\boxed{\varphi_{\pm}(t) = (\pm 1, \sqrt{3}\cos t, \sqrt{3}\sin t)}$$

Problem 9 [10pts] Suppose $G(x, y, z) = (xy, yz)$. Answer the following questions without solving any equations. Instead, use a theorem to justify your claims:

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(a.) where is it possible to locally solve $G(x, y, z) = (1, 1)$ for y, z as functions of x

Implicit Function Th^m { Observe G is continuously diff near all points. Thus we need $\det\left[\frac{\partial G}{\partial y}\Big|\frac{\partial G}{\partial z}\right] = \det\begin{bmatrix} x & 0 \\ z & y \end{bmatrix} = xy \neq 0$.

Any point for which $G(P) = (1,1)$ and $P_x P_y \neq 0$ will do.

(b.) where is it possible to locally solve $G(x, y, z) = (1, 1)$ for x, z as functions of y

Again, apply implicit function theorem.

$\det\left[\frac{\partial G}{\partial x}\Big|\frac{\partial G}{\partial z}\right] = \det\begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = y^2 \neq 0$

Thus $P = (P_1, P_2, P_3)$ for which $G(P) = (1,1)$ and $P_2^2 \neq 0$ will do.

Problem 10 [10pts] Let $F(x, y, z) = (x+y, x^2+y^2, x^3+y^3+z^3)$. Calculate $F'(x, y, z)$ and find where F is locally invertible.

$$F = \begin{bmatrix} x+y \\ x^2+y^2 \\ x^3+y^3+z^3 \end{bmatrix} \longrightarrow F'(x,y,z) = \begin{bmatrix} 1 & 1 & 0 \\ 2x & 2y & 0 \\ 3x^2 & 3y^2 & 3z^2 \end{bmatrix}$$

$$\det(F'(x,y,z)) = 3z^2(1(2y) - 1(2x))$$
$$= \underline{6z^2(y-x)}$$

Thus, avoid $z = 0$ and $y = x$. Away from these planes we have local invertibility by inv. fnct. Th^m.

Problem 11 [10pts] Find extrema of $f(x, y) = 2x^2 + 4y^2$ on the unit-circle $x^2 + y^2 = 1$.

$$\nabla f = \langle 4x, 8y \rangle = \lambda \langle 2x, 2y \rangle$$

$$\left.\begin{array}{l} 4x = \lambda(2x) \Rightarrow 4xy = 2\lambda xy \\ 8y = \lambda(2y) \Rightarrow 8xy = 2\lambda xy \end{array}\right\} \Rightarrow xy = 0$$

Thus either $x = 0$ or $y = 0$ as $x=0, y=0$ forbidden by $x^2+y^2=1$ it follows $(\pm 1, 0)$ and $(0, \pm 1)$ are extremal points.

In particular, $f(\pm 1, 0) = \underbrace{2}_{\text{min.}}$ and $f(0, \pm 1) = \underbrace{4}_{\text{max.}}$

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Problem 12 [10pts] Let $\Phi(t) = (t, t^2, t^3, t^4)$ for all $t \in \mathbb{R}$. Define $C = \Phi(\mathbb{R})$. Let $p = (1,1,1,1)$. Derive the tangent and normal spaces to C at p; that is calculate T_pC and N_pC. You may describe T_pC and N_pC as a span or as point-sets in $\mathbb{R}^4$ given by cartesian equations, your choice.

$$\Phi'(t) = \langle 1, 2t, 3t^2, 4t^3 \rangle$$

$$\Phi'(1) = \langle 1, 2, 3, 4 \rangle \qquad p = (1,1,1,1) = \Phi(1)$$

Thus $\boxed{T_pC = p + \text{span}\{(1,2,3,4)\}}$

$$N_pC = \{ (x,y,z,t) \in \mathbb{R}^4 \mid \langle x-1, y-1, z-1, t-1 \rangle \cdot \langle 1,2,3,4 \rangle = 0 \}$$

$$= F^{-1}(\{0\}) \text{ for } \underline{F(x,y,z,t) = x-1 + 2(y-1) + 3(z-1) + 4(t-1)}.$$

Problem 13 [10pts] Suppose functions of the form $f : \mathbb{R}^3 \to \mathbb{R}$ have multivariate power series expansions centered at p as given below. In each case, identify $\nabla f(p)$ and decide if p is a critical point. If p is a critical point then use the theory of quadratic forms to classify the extrema.

(a.) $p = (1,2,3)$,

$$f(x,y,z) = 10 + (x-1) + 2(z-3) + (x-1)^2 + (y-2)^2 + \cdots$$

$$\nabla f(1,2,3) = \langle 1, 0, 2 \rangle \text{ : not critical}$$

(b.) $p = (1,2,4)$,

$$f(x,y,z) = 3 - (x-1)^2 - (y-2)^2 - 4(z-4)^2 + \cdots$$

$$\nabla f(1,2,4) = \langle 0,0,0 \rangle \text{ : critical.}$$

Note $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = -4 \Rightarrow \underline{f(1,2,4) = 3}$ local max.

(c.) $p = (0,0,0)$, hint, try $\lambda = -1$.

$$f(x,y,z) = 3 + 2xy + 2xz + 2yz \cdots$$

$$\det(Q - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda)$$

$$= -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= (\lambda+1)(-\lambda^2 + \lambda + 2)$$

$$= -(\lambda+1)(\lambda-2)$$

$\lambda_1 = \lambda_2 = -1$

$\lambda_3 = 2$

$\therefore f(0,0,0)$ saddle pt.

$$\begin{array}{r} -\lambda^2 + \lambda + 2 \\ \lambda + 1 \overline{\smash{\big)} -\lambda^3 + 3\lambda + 2} \\ \underline{-\lambda^3 - \lambda^2} \\ \lambda^2 + 3\lambda + 2 \end{array}$$

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Problem 14 [10pts] Let V, W be finite dimensional vector spaces with norms $||\cdot||_V$ and $||\cdot||_W$ respectively. Assume F, G are differentiable functions from V to W which are (Frechet) differentiable at p. Show that $F+G$ is also differentiable at p.

Consider, dF_p, dG_p exist and are linear by assumption. Claim $dF_p + dG_p = d(F+G)_p$, consider,

$$\lim_{H \to 0} \left(\frac{F(P+H) + G(P+H) - F(P) - G(P) - d(F+G)_P}{\|H\|} \right) =$$

$$= \lim_{H \to 0} \frac{F(P+H) - F(P) - dF_P}{\|H\|} + \lim_{H \to 0} \frac{G(P+H) - G(P) - dG_P}{\|H\|}$$

$$= 0 + 0$$

$$= 0.$$

Thus $F+G$ is diff at p with $\underbrace{d(F+G)_p}_{\text{know this is linear}} = dF_p + dG_p$.

since sum of lin. trans. is linear.

Problem 15 [10pts] Let $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $G(x, y) = x^T J y$ for some invertible matrix J. Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has $T(x) = Ax$ for all $x \in \mathbb{R}^n$. If $G(T(x), T(y)) = G(x, y)$ for all $x, y \in \mathbb{R}^n$ then what condition does this force A to satisfy?

$$G(T(x), T(y)) = G(x, y) \quad \forall x, y \in \mathbb{R}^n$$

$$\Rightarrow (Ax)^T J Ay = x^T J y \quad \forall x, y \in \mathbb{R}^n$$

$$\Rightarrow x^T A^T J A y = x^T J y \quad \forall x, y \in \mathbb{R}^n$$

$$\Rightarrow \boxed{A^T J A = J}$$

Problem 16 [10pts] Calculate $dG(v, w)$ for G given in the previous problem. ~~I believe the formula for dG is independent of the point at which dG is calculated.~~

$$G(V+H, W+K) = (V+H) J (W+K)$$

$$= \underbrace{VJW}_{G(V,W)} + \underbrace{VJK + HJW}_{dG(V,W)(H,K)} + \underbrace{HJK}_{}$$

2nd order will die when we take the Frechet Limit

$$\Rightarrow \boxed{(dG_{(V,W)})(H,K) = VJK + HJW}$$