Problem 1 (15pts) Let $F(x,y) = (x^2+y, x+y^2, xy) = (u,v,w)$. Let $\alpha = u^2 dv \wedge dw$ and $\beta = du \wedge dv \wedge dw$. Calculate $F^*\alpha$ and $F^*\beta$. Hint: one of these is really easy.

Problem 2 (15pts) Let $\alpha = (x^2+y^2)dz$ and $\beta = ydz + x^2dy$.

(a.) calculate $\alpha \wedge \beta$

(b.) calculate $d\alpha$ and $d\beta$

(c.) calculate $d\alpha \wedge \beta - \alpha \wedge d\beta$

(d.) calculate $d(\alpha \wedge \beta)$

Problem 3 (30pts) Let $\omega_{\langle a,b,c\rangle} = adx + bdy + cdz$ and $\Phi_{\langle a,b,c\rangle} = a\, dy \wedge dz + b\, dz \wedge dx + c\, dx \wedge dy$ define the **work** and **flux** form mappings where a,b,c are generally functions on $\mathbb{R}^3$. In another notation, if $\vec{F} = \langle F^1, F^2, F^3\rangle$ then

$$\omega_{\vec{F}} = \sum_{i=1}^{3} F^i dx^i \qquad \& \qquad \Phi_{\vec{F}} = \frac{1}{2}\sum_{i,j,k=1}^{3} \epsilon_{ijk} F^i dx^j \wedge dx^k$$

Work out the following identities: assume h is a real-valued function and $\vec{D}, \vec{E}, \vec{F}, \vec{G}$ are vector fields,

(a.) show the formulas for $\Phi_{\vec{F}}$ are consistent.

(b.) $\omega_{\vec{F}+h\vec{G}} = \omega_{\vec{F}} + h\Phi_{\vec{G}}$

(c.) $\Phi_{\vec{F}+h\vec{G}} = \Phi_{\vec{F}} + h\Phi_{\vec{G}}$

(d.) $\omega_{\vec{F}} \wedge \omega_{\vec{G}} = \Phi_{\vec{F}\times\vec{G}}$

(e.) $\omega_{\vec{E}} \wedge \Phi_{\vec{D}}$

(f.) $\omega_{\vec{E}} \wedge \omega_{\vec{F}} \wedge \omega_{\vec{G}}$

Problem 4 (15pts) Continuing the previous problem, next derive the differential identities below: for h a function and $\vec{F}, \vec{G}$ vector fields (all differentiable)

(a.) $dh = \omega_{\nabla h}$

(b.) $d\omega_{\vec{F}} = \Phi_{\nabla\times\vec{F}}$

(c.) $d\Phi_{\vec{G}} = (\nabla \bullet \vec{G})\, dx \wedge dy \wedge dz$

Problem 5 (15pts) We derived in Lecture the product rule for the exterior derivative of differential forms α and β is $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ where α is a p-form. Given this product rule and the identities already derived in the previous problems, show that:

(a.) $d\omega_{f\vec{G}} = d\left(f\omega_{\vec{G}}\right)$ implies $\nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f\nabla \times \vec{G}$

(b.) $d\Phi_{f\vec{G}} = d\left(f\Phi_{\vec{G}}\right)$ implies $\nabla \bullet (f\vec{G}) = \nabla f \bullet \vec{G} + f\nabla \bullet \vec{G}$

(c.) $d\Phi_{\vec{F}\times\vec{G}} = d(\omega_{\vec{F}} \wedge \omega_{\vec{G}})$ implies $\nabla \bullet (\vec{F}\times\vec{G}) = (\nabla \times \vec{F}) \bullet \vec{G} - \vec{F} \bullet (\nabla \times \vec{G})$

Problem 6 (20pts) Hodge duality allows us to access additional identities of vector calculus. We can summarize Hodge duality for Euclidean three dimensional space with respect to vol $= dx \wedge dy \wedge dz$ by:

$$\star\omega_{\vec{F}} = \Phi_{\vec{F}}, \quad \star\Phi_{\vec{F}} = \omega_{\vec{F}}, \quad \star f\, dx \wedge dy \wedge dz = f, \quad \star f = f dx \wedge dy \wedge dz.$$

Hodge duality is an isomorphism of p and $(n-p)$ forms which is function linear much like the work and flux form mappings. Let $\mathbb{D}^2$ act on p-forms according to the rule below:

$$\mathbb{D}^2 = (-1)^{p+1}(d \star d \star \ - \ \star d \star d).$$

Derive formulas for $\mathbb{D}^2$ for $p = 0, 1, 2$ and 3. For maximum transparency make use of the identity $\nabla^2 \vec{F} = \nabla(\nabla \bullet \vec{F}) - \nabla \times (\nabla \times \vec{F})$ for the $p = 1$ and $p = 2$ cases. As a slight abuse of notation, we might write $\mathbb{D}^2 =$ ________.

Problem 7 (10pts) Given a vector field X and a differential one-form α we define

$$X \lrcorner\, \alpha = \alpha(X)$$

Likewise, for a p-form γ we define a $(p-1)$-form $X \lrcorner\, \gamma$ by

$$X \lrcorner\, \gamma = \gamma(X, -, \dots, -) \qquad \text{meaning} \qquad (X \lrcorner\, \gamma)(v_1, \dots, v_{p-1}) = \gamma(X, v_1, \dots, v_{p-1})$$

If $X = \sum_{i=1}^{n} X^i \partial_i$ and $\alpha = \sum_{j=1}^{n} \alpha_j dx^j$ and $\beta = \sum_{i<j} \beta_{ij} dx^i \wedge dx^j$ where $dx^j(\partial_i) = \delta_{ij}$ then calculate $X \lrcorner\, \alpha$ and $Y \lrcorner\, X \lrcorner\, \beta$ in terms of sums of products of the coefficients $X^i, \alpha_j, \beta_{ij}$. Also, supposing we have a metric with which we can construct the musical morphisms, what is $\vec{A} \lrcorner\, (\flat \vec{B})$

Problem 8 (20pts) (this is a play on P103 and P104 of 2013) The volume form for a hypersurface in $\mathbb{R}^n$ has a fairly easy form to compute. In particular, it can be shown for $M = f^{-1}\{c\}$ the volume form on M is implicitly given by

$$df \wedge \text{vol}(M) = ||\nabla f|| dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

where we assume $\nabla f \neq 0$ on M. Calculate the volume of the 3-dimensional subset M of $\mathbb{R}^4$ defined by $f(x, y, z, v) = x^2 + y^2 = 4$ for $0 \leq z, v \leq 1$. Suggestion, notice:

$$dx \wedge dy = r dr \wedge d\theta$$

is still relevant and in hypercylindrical coordinates (r, θ, z, v) the volume M is described by $f = r^2 = 4$ where $0 \leq \theta \leq 2\pi$ and $0 \leq z, v \leq 1$.

Problem 9 (20pts) Consider the solid tesseract $M = [0, 1]^4 \subset \mathbb{R}^4_{xyzw}$. The boundary ∂M consists of 8-three-dimensional faces which stem from $x, y, z, w = 0, 1$. Calculate:

$$\int_{\partial M} \left[(-12wx + y\tan(z)) dx \wedge dy \wedge dz + (\cos^3(yz^2w^3) + 2x^2w) dy \wedge dz \wedge dw\right]$$

If you try to do this directly, then Loki wins. See past the hideous face of the problem, it's really quite easy.

Problem 10 (10pts) Recall, a differential equation $\omega = Mdx + Ndy = 0$ is said to be **exact** if there exists F for which $dF = \omega$. Moreover, if the equation is exact then the solutions are simply level curves of F. Notice $d\omega = 0$ is a necessary consequence of the general identity $d^2 = 0$ applied to $d(dF) = 0$. Further, by Poincare's Lemma if the domain of the DEqn in contractible then $d\omega = 0$ implies $\omega = dF$ for some function F. Consider:

$$(x^2 + yz)dx + (y^3 + xz)dy + (z^4 + xy)dz = 0$$

can you solve this differential equation? Is it exact? The solution ought to have the form $F(x, y, z) = C$ for an appropriate level-function F

Problem 11 (15pts) Consider the Faraday two-form $F = dt \wedge \omega_{\vec{E}} + \Phi_{\vec{B}}$ given as:

$$F = dt \wedge (t + x)dx + ydt \wedge dy + y^2 dy \wedge dz + (z - y)dz \wedge dx + zdx \wedge dy$$

Find $\vec{E}$ and $\vec{B}$ implicit within the given F. Also, find a potential one-form A for which $dA = F$.

Problem 12 (15pts, a.k.a. Problem 146) Let $G(x, y, z, w) = (x^2 + y^2,\ zw,\ y + z)$. Define $M = G^{-1}\{(25, 1, 0)\}$. Find the tangent and normal spaces to M at the point $(3, 4, -4, -1/4)$.