Problem 91 We define the coderivative $\delta : \Lambda^p M \to \Lambda^{p-1} M$ by:

$$\delta\alpha = (-1)^{np+n+1} \star d \star \alpha$$

Observe $\delta\alpha$ is a differential form of one-less degree than α. In contrast, $d\alpha$ is a differential form of one-more degree than α. We proved, or observed, that $d^2 = 0$ and $\star\star = \pm 1$ where the sign depends both on the signature of the metric on M as well as the degree of the form and the dimension of M. However, those details need not concern us for the following question: **Show:**

$$\delta^2 \alpha = 0$$

Problem 92 In the previous problem I was intentionally vague about which manifold and metric we were using. Here, I wil be precise: let us consider $M = \mathbb{R}^3$ with the Euclidean metric. Find nice formulas for: here $\omega_{\vec{F}} = Pdx + Qdy + Rdz$ as usual,

(a.) $d\delta\omega_{\vec{F}}$ where $\vec{F} = \langle P, Q, R \rangle$.

(b.) $\delta d\omega_{\vec{F}}$ where $\vec{F} = \langle P, Q, R \rangle$.

(c.) is $d\delta + \delta d$ anything interesting?

Problem 93 Suppose M is a manifold with boundary ∂M and suppose α, β are differential forms on M with $deg(\alpha) + deg(\beta) = dim(M) - 1$. Relate $\int_M \alpha \wedge d\beta$ and $\int_M \beta \wedge d\alpha$; that is, derive integration-by-parts for differential forms.

Problem 94 In $\mathbb{R}^4$ with metric $\eta = Diag(-1, 1, 1, 1)$ I describe in my notes how Hodge duality introduces certain signs. The basic idea is very much like the simpler context of $\mathbb{R}^3$. Because I use the metric which agrees with the euclidean metric on x, y, z components the work and flux-form correspondences naturally generalize: a general one-form on $\mathbb{R}^4_{txyz}$ space has the form:

$$\alpha = \alpha_0 dt + \alpha_1 dx + \alpha_2 dy + \alpha_3 dz = \alpha_0 dt + \omega_{\vec{\alpha}}.$$

Notice, I am encouraging the notation $\vec{\alpha}$ for the spatial vector piece of the one-form α. No such simple correspondence is possible for a generic two-form since it has six independent components:

$$\begin{aligned}\beta &= F_1 dt \wedge dx + F_2 dt \wedge dy + F_3 dt \wedge dz + G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy \\ &= dt \wedge \omega_{\vec{F}} + \Phi_{\vec{G}}\end{aligned}$$

Clearly the formula in terms of the work and flux-form correspondance will make it easier for us to follow calculus and algebra for β. Next, a three-form has the general form:

$$\begin{aligned}\gamma &= G_0 dx \wedge dy \wedge dz + G_1 dt \wedge dy \wedge dz + G_2 dt \wedge dz \wedge dx + G_3 dt \wedge dx \wedge dy \\ &= G_0 dx \wedge dy \wedge dz + dt \wedge \Phi_{\vec{\gamma}}\end{aligned}$$

where I am encouraging use of the notation $\vec{\gamma} = \langle G_1, G_2, G_3 \rangle$ to emphasize the correspondence between spatial 3-vectors and those components of γ. Continuting, there is just one 4-form:

$$\zeta = f dt \wedge dx \wedge dy \wedge dz.$$

Please notice that all the coefficients of the forms are in fact 0-forms on $\mathbb{R}^4$, that is, functions of t, x, y, z. This introduces time derivative terms in the formulas you are to find below. Use the notation given above to calculate:

(a.) df where f is a real-valued function on $\mathbb{R}^4_{txyz}$.

(b.) $d\alpha$

(c.) $d\beta$

(d.) $d\gamma$

(e.) $d\zeta$

Problem 95 Again, using the notation introduced in the previous problem, find the explicit (and as nice as possible) formulas for: (use the results on page 221 of my notes)

(a.) $\star f$ where f is a real-valued function on $\mathbb{R}^4_{txyz}$.

(b.) $\star\alpha$

(c.) $\star\beta$

(d.) $\star\gamma$

(e.) $\star\zeta$

Problem 96 Derive differential identities from $d^2 = 0$ on $\mathbb{R}^4_{txyz}$. In other words, work out the analog of Problem 80 for four-dimensional Minkowski space.

Problem 97 Suppose $L(y, \dot{y}, t) = y^3 + y\dot{y}^2$ is the Lagrangian for some model. Find the Euler-Lagrange equations for y.

Problem 98 The three-dimensional free-particle Lagrangian for mass m is simply

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right].$$

We saw in lecture the Euler-Lagrange equations for x, y, z were simply:

$$m\ddot{x} = 0, \qquad m\ddot{y} = 0, \qquad m\ddot{z} = 0.$$

Find the L in spherical coordinates:

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi$$

and derive the Euler Lagrange equations for r, θ, ϕ.

Problem 99 Find the geodesic equations for the paraboloid $z = x^2 + y^2$. If it is within your power, solve them. (I'm not sure how horrible this suggestion is, but where effort is made, credit will likely flow)

Problem 100 Follow link *http://mathoverflow.net/q/26939/24854*. Read page. We're on the honor system here, I just want you to scan through it to get a feel for how professional mathematicians think about forms.

Problem 101 Follow link *http://mathoverflow.net/q/10574/24854*. Read page. We're on the honor system here, I just want you to scan through it to get a feel for how professional mathematicians think about forms. (if your time is limited then your time is better spend on this thread, this gives a lot of ideas for future directions and reading on differential forms)

Problem 102 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with non-vanishing ∇f. Let M be the hypersurface which is formed by the solution set of $f(x) = c$; that is $M = f^{-1}\{c\}$. Furthermore, let $n = \omega_{\vec{n}}$ be **unit-normal form** in the sense that $\text{Ker}(n_p) = T_pM$ for each $p \in M$ and $\vec{n} \bullet \vec{n} = 1$. We define the volume form vol_M on the hypersurface by $\text{vol}_M = \star n$ where $\star$ is the eulicidean Hodge dual. Show:

$$df \wedge \text{vol}_M = |\nabla f| dx^1 \wedge \cdots \wedge dx^n$$

Problem 103 Calculate, use the volume form defined in previous problem,

$$\int_{S_R} \text{vol}_M = 2\pi R$$

where $S_R = F^{-1}(R)$ for $F(x, y) = x^2 + y^2$. Suggestion,

$$n = xdy - ydx$$

has $\vec{n} = \langle -y, x\rangle$ with unit-length on S_R.

Problem 104 Calculate, use the volume form defined in previous problem,

$$\int_{S_R} \text{vol}_M = 4\pi R^2$$

where $S_R = F^{-1}(R)$ for $F(x, y, z) = x^2 + y^2 + z^2$.

Problem 105 Consider the 1-form $\alpha = xdz + ydw - (x^2 + y^2 + z^2 + w^2)dt$ on $\mathbb{R}^5$. Calculate $\int_S d\alpha \wedge d\alpha$, where $S \subset \mathbb{R}^5$ is given by $x^2 + y^2 + z^2 + w^2 = 1$ and $0 \leq t \leq 1$. Use the generalized Stokes' Theorem and the identity $d\alpha \wedge d\alpha = d(\alpha \wedge d\alpha)$ to make life easier.

Problem 106 The rank of a one-form ω is defined, at a point, to be the largest positive integer r for which $\omega_r \neq 0$ yet $\omega_{r+1} = 0$ where we define the **auxillary forms** $\omega_1, \omega_2, \ldots$ as follows:

$$\omega_1 = \omega,\ \omega_2 = d\omega,\ \omega_3 = \omega \wedge d\omega,\ \omega_4 = d\omega \wedge d\omega,\ \omega_5 = \omega \wedge d\omega \wedge d\omega, \ldots$$

The auxillary forms $\{\omega_1, \ldots, \omega_r\}$ form a LI set of forms in the exterior algebra of the manifold at a the given point. If the rank of the one-form is constant at all points then the form is called **regular**.Find the rank of:

$$\omega = (x + y)dt + ydz$$

on $\mathbb{R}^4_{txyz}$ space.

Problem 107 Consider the one-form $\omega = xdx + ydy + zdz$ on $\mathbb{R}^3$. Find the foliation of three dimensional space into two-dimensional submanifolds whose tangent spaces are spanned by vector fields which are found in $\ker(\omega)$. Check the condition needed to show ω is dual to a two-plane field distribution on $\mathbb{R}^3$; that is verify $\omega \wedge d\omega = 0$ (see Bachman for where I'm coming from here). Incidentally, there is one point left out, perhaps it would be more honest to say find a foliation of $\mathbb{R}^3 - \{(0, 0, 0)\}$.

Problem 108 Consider $\omega = dy + dz + xydx + xzdx$. Show that $\omega \wedge d\omega = 0$ on all of $\mathbb{R}^3$. What foliation of $\mathbb{R}^3$ does ω describe. Recall, we discussed that $\omega = dz$ corresponds to foliating $\mathbb{R}^3$ into $z = c$ (a family of horizontal planes, each leaf in the foliation labeled by c). Try to find the corresponding family of surfaces for the ω given here.

Problem 109 Prove Cartan's Lemma. This is cleanly stated as Exercise 1.33 on page 26 of in the *Equivalence, Invariance and Symmetry* the relevant portion of which is posted as a pdf I posted in Course Content.

Problem 110 Now that you understand a bit about the wedge product: challenge: define the wedge product in terms of the determinant. Start by experimenting in $n = 2$ or $n = 3$ to see what to do in general for k-forms in $\mathbb{R}^n$.