# MATH 332, ADVANCED CALCULUS, TEST 3 SOLUTION

1.) Let $\alpha, \beta \in V^*$ where $V$ is finite dim'l v-space over $\mathbb{R}$

(a.)
$$\begin{aligned}
(\alpha \otimes \beta)(cx+y, z) &= \alpha(cx+y)\beta(z) && : \text{def}^{\underline{n}} \text{ of } \otimes \\
&= (c\,\alpha(x) + \alpha(y))\beta(z) && : \alpha \in V^* \text{ is linear.} \\
&= c\,\alpha(x)\beta(z) + \alpha(y)\beta(z) && : \text{algebra} \\
&= c(\alpha \otimes \beta)(x,z) + (\alpha \otimes \beta)(y,z) && : \otimes \text{ def}^{\underline{n}} \text{ once more.}
\end{aligned}$$

A similar calculation shows:

$$(\alpha \otimes \beta)(x, cy+z) = c(\alpha \otimes \beta)(x,y) + (\alpha \otimes \beta)(x,z)$$

Thus $\alpha \otimes \beta : V \times V \longrightarrow \mathbb{R}$ is bilinear.

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(b.)
$$\begin{aligned}
(\alpha \wedge \beta)(x,y) &= (\alpha \otimes \beta - \beta \otimes \alpha)(x,y) \\
&= \alpha(x)\beta(y) - \beta(x)\alpha(y) \\
&= -(\beta(x)\alpha(y) - \alpha(x)\beta(y)) \\
&= -(\beta \otimes \alpha - \alpha \otimes \beta)(x,y) \\
&= -(\beta \wedge \alpha)(x,y) \quad \Rightarrow \quad \underline{\alpha \wedge \beta = -\beta \wedge \alpha}.
\end{aligned}$$

Alternatively,

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha = -(\beta \otimes \alpha - \alpha \otimes \beta) = -\beta \wedge \alpha.$$

2.) Let $\omega_{\vec F}$ and $\Phi_{\vec G}$ be defined as usual

(a.) $$\omega_{\vec F}\wedge\omega_{\vec G} = (F_1\,dx + F_2\,dy + F_3\,dz)\wedge(G_1\,dx + G_2\,dy + G_3\,dz)$$
$$= F_1G_2\,dx\wedge dy + F_1G_3\,dx\wedge dz + F_2G_1\,dy\wedge dz + F_2G_3\,dy\wedge dz + F_3G_1\,dz\wedge dx + F_3G_2\,dz\wedge dy$$
$$= \underbrace{(F_1G_2 - F_2G_1)}_{(\vec F\times\vec G)_3}dx\wedge dy + \underbrace{(F_3G_1 - F_1G_3)}_{(\vec F\times\vec G)_2}dz\wedge dx + \underbrace{(F_2G_3 - F_3G_2)}_{(\vec F\times\vec G)_1}dy\wedge dz$$
$$= \Phi_{\vec F\times\vec G}\,.$$

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(b.) $$\omega_{\vec A}\wedge\omega_{\vec B}\wedge\omega_{\vec C} = \Phi_{\vec A\times\vec B}\wedge\omega_{\vec C}$$
$$= \left[(\vec A\times\vec B)_1\,dy\wedge dz + (\vec A\times\vec B)_2\,dz\wedge dx + (\vec A\times\vec B)_3\,dx\wedge dy\right]\wedge\left[C_1\,dx + C_2\,dy + C_3\,dz\right]$$
$$= (\vec A\times\vec B)_1C_1\,dy\wedge dz\wedge dx + (\vec A\times\vec B)_2C_2\,dz\wedge dx\wedge dy + (\vec A\times\vec B)_3C_3\,dx\wedge dy\wedge dz$$
$$= \left[(\vec A\times\vec B)_1C_1 + (\vec A\times\vec B)_2C_2 + (\vec A\times\vec B)_3C_3\right]dx\wedge dy\wedge dz$$
$$= \left[(\vec A\times\vec B)\cdot\vec C\right]dx\wedge dy\wedge dz\,.$$

Well, of course!, $\underline{\omega_{\vec A}\wedge\omega_{\vec B}\wedge\omega_{\vec C} = \det(\vec A|\vec B|\vec C)\,dx\wedge dy\wedge dz}$ and we know, $\det(\vec A|\vec B|\vec C) = (\vec A\times\vec B)\cdot\vec C$.

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(c.) $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \omega_{\nabla f}\,.$$

You probably solved (d.) less strangely than I,

(d.) $$d\omega_{\vec F} = dF_1\wedge dx + dF_2\wedge dy + dF_3\wedge dz\,,\quad \vec F = \langle F_1, F_2, F_3\rangle$$
$$= \omega_{\nabla F_1}\wedge\omega_{\hat x} + \omega_{\nabla F_2}\wedge\omega_{\hat y} + \omega_{\nabla F_3}\wedge\omega_{\hat z}$$
$$= \Phi_{\nabla F_1\times\hat x} + \Phi_{\nabla F_2\times\hat y} + \Phi_{\nabla F_3\times\hat z}$$
$$= \Phi_{\nabla F_1\times\hat x + \nabla F_2\times\hat y + \nabla F_3\times\hat z} = \Phi_{\nabla\times\vec F}$$

3.)

$$\alpha \wedge \beta = \left(\sum_{i_1 \cdots i_p} \alpha_{i_1 \cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}\right) \wedge \left(\sum_{j_1 \cdots j_q} \beta_{j_1 \cdots j_q}\, dx^{j_1} \wedge \cdots \wedge dx^{j_q}\right)$$

$$= \sum_{i_1 \cdots i_p} \sum_{j_1 \cdots j_q} \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q} (-1)^p\, dx^{j_1} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_q}$$

$$= \sum_I \sum_J \alpha_I \beta_J (-1)^p (-1)^p\, dx^{j_1} \wedge dx^{j_2} \wedge dx^I \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_q}$$

$\vdots$

$$= \sum_I \sum_J \alpha_I \beta_J \underbrace{(-1)^p (-1)^p \cdots (-1)^p}_{q}\, dx^J \wedge dx^I$$

$$= (-1)^{pq} \left(\sum_{j_1 \cdots j_q} \beta_{j_1 \cdots j_q}\, dx^{j_1} \wedge \cdots \wedge dx^{j_q}\right) \wedge \left(\sum_{i_1 \cdots i_p} \alpha_{i_1 \cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}\right)$$

$$= \underline{(-1)^{pq}\, \beta \wedge \alpha}.$$

4.) Given $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$ calculate

(a.) $d(\omega_{\vec F} \wedge \omega_{\vec G})$, $\vec F$ & $\vec G$ a vector field

(b.) $d(f \omega_{\vec G})$, $f$ a fnct., $\vec G$ a vector field.

(c.) $d(f\, \Phi_{\vec G})$, $f$ a fnct, $\vec G$ a vector field.

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(a.) $d(\omega_{\vec F} \wedge \omega_{\vec G}) = d\omega_{\vec F} \wedge \omega_{\vec G} - \omega_{\vec F} \wedge d\omega_{\vec G}$

$$\Rightarrow d\,\Phi_{\vec F \times \vec G} = d\omega_{\vec F} \wedge \omega_{\vec G} - \omega_{\vec F} \wedge d\omega_{\vec G}$$

$$\Rightarrow (\nabla \cdot (\vec F \times \vec G))\, dx \wedge dy \wedge dz = \Phi_{\nabla \times \vec F} \wedge \omega_{\vec G} - \omega_{\vec F} \wedge \Phi_{\nabla \times \vec G}$$

$$= [(\nabla \times \vec F) \cdot \vec G - \vec F \cdot (\nabla \times \vec G)]\, dx \wedge dy \wedge dz$$

$$\therefore \boxed{\nabla \cdot (\vec F \times \vec G) = (\nabla \times \vec F) \cdot \vec G - \vec F \cdot (\nabla \times \vec G)} \quad \text{neat.}$$

4 continued

(b.) $d(f\omega_{\vec{G}}) = df \wedge \omega_{\vec{G}} + f\, d\omega_{\vec{G}}$

$= \omega_{\nabla f} \wedge \omega_{\vec{G}} + f\, \Phi_{\nabla \times \vec{G}}$

$= \Phi_{\nabla f \times \vec{G}} + \Phi_{f(\nabla \times \vec{G})}$

$= \Phi_{\nabla f \times \vec{G} + f(\nabla \times \vec{G})}$

But,

$$d(f\omega_{\vec{G}}) = d(\omega_{f\vec{G}}) = \Phi_{\nabla \times (f\vec{G})}$$

Hence, we derive,

$$\boxed{\nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f\, \nabla \times \vec{G}}$$

Remark: I'm happy to see the $(-1)^{\deg(\alpha)}$ term generating all the weird signs in these identities.

(c.) $d(f\Phi_G) = df \wedge \Phi_{\vec{G}} + f\, d\Phi_{\vec{G}}$

$= \omega_{\nabla f} \wedge \Phi_{\vec{G}} + f(\nabla \cdot \vec{G})\, dx \wedge dy \wedge dz$

$= \Phi_{\vec{G}} \wedge \omega_{\nabla f} + f(\nabla \cdot \vec{G})\, dx \wedge dy \wedge dz$

$= [\vec{G} \cdot \nabla f + f\, \nabla \cdot \vec{G}]\, dx \wedge dy \wedge dz$

However, we also have:

$$d(f\,\Phi_{\vec{G}}) = d(\Phi_{f\vec{G}}) = [\nabla \cdot (f\vec{G})]\, dx \wedge dy \wedge dz$$

$$\therefore \boxed{\nabla \cdot (f\vec{G}) = f(\nabla \cdot \vec{G}) + (\nabla f) \cdot \vec{G}}$$

6.) $F = dt \wedge \omega_{\vec{E}} + \Phi_{\vec{B}} = dA$

Given $A = (x^2+y^2)\,dt - t\,dx$, find $\vec{E}$ & $\vec{B}$

$$dA = (2x\,dx + 2y\,dy)\wedge dt - dt \wedge dx$$
$$= dt \wedge \left[-(2x+1)\,dx - 2y\,dy\right] + \Phi_{\vec{0}} = dt \wedge \omega_{\vec{E}} + \Phi_{\vec{B}}$$

Comparing we find, by LI for forms with $dx\wedge dy$, $dy \wedge dz$, $dz\wedge dx$, ~~dtnd~~ $dt\wedge dx$, $dt\wedge dy$, $dt\wedge dz$ the coeff. of these terms must be LI hence we can equate coeff. and read:

$$\boxed{\vec{E} = \langle -2x+1, -2y, 0\rangle, \quad \vec{B} = \langle 0,0,0\rangle}$$

If you want to generate $\vec{B} \neq 0$ then we'll need to include terms like $x\,dy$ or $y\,dz$ etc.. in the one-form potential.