Please put your work on this page. Box your answers. Thanks and enjoy. You are allowed use of the Laplace transform sheet and your notes today.

Problem 1 [10pts] Solve $y'' + 6y' + 10y = \delta(t-3)$ where $y'(0) = 1$ and $y(0) = 0$.

$$s^2 Y - s\cancel{y(0)}^{0} - y'(0) + 6(sY - \cancel{y(0)}^{0}) + 10Y = e^{-3s}$$

$$(s^2 + 6s + 10)Y = 1 + e^{-3s}$$

$$Y(s) = \frac{1}{(s+3)^2+1} + \left(\frac{1}{(s+3)^2+1}\right)e^{-3s}$$

$$\boxed{y(t) = e^{-3t}\sin t + e^{-3(t-3)}\sin(t-3)\,u(t-3)}$$

Problem 2 [10pts] Solve $y'' + 9y = f(t)$ where $y(0) = 1$, $y'(0) = 1$ and $f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

$$f(t) = t\,[u(t) - u(t-1)] = t\,u(t) - t\,u(t-1)$$

$$\mathcal{L}\{t\,u(t)\}(s) = \mathcal{L}\{t\}\,e^{-0\cdot s} = \frac{1}{s^2}$$

$$\mathcal{L}\{t\,u(t-1)\}(s) = \mathcal{L}\{t+1\}(s)\,e^{-s} = \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s}$$

Thus, the Laplace transform of $y'' + 9y = f(t)$ is,

$$s^2\,Y + 9Y - s - 1 = \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s}$$

$$(s^2 + 9)\,Y = s + 1 + \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s}$$

$$Y(s) = \underbrace{\frac{s+1}{s^2+9} + \frac{1}{s^2(s^2+9)}}_{F_1 + F_2} - \underbrace{\frac{1}{s^2+9}\left(\frac{1}{s^2} + \frac{1}{s}\right)}_{F_3} e^{-s}$$

$$F_2 = \frac{1}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2+9} = \frac{+1}{9s^2} - \frac{1}{9(s^2+9)} = F_2 \quad \text{I}$$

$$\boxed{\begin{aligned} 1 &= As(s^2+9) + B(s^2+9) + (Cs+D)s^2 \\ 1 &= s^3(A+C) + s^2(B+D) + s(9A) + 9B \\ &\text{Equate Coeff.}\quad B = 1/9,\quad 9A = 0,\quad B + D = 0,\quad A + C = 0 \\ &\text{Hence } D = -1/9,\ A = 0,\ C = 0 \end{aligned}}$$

$$F_3 = \frac{1}{s^2+9}\left(\frac{1}{s^2} + \frac{1}{s}\right) = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2+9} = \frac{1}{9}\left(\frac{1}{s} + \frac{1}{s^2} - \frac{s+1}{s^2+9}\right) = F_3 \quad \text{II}$$

$$\boxed{\begin{aligned} \frac{(s^2+9)}{s^2+9}\left(\frac{s^2}{s^2} + \frac{s^2}{s}\right) &= (As + B)(s^2+9) + (Cs+D)s^2 \\ 1 + s &= (As + B)(s^2 + 9) + s^2(Cs + D) \\ &= s^3(A + C) + s^2(B + D) + s(9A) + 9B \\ &\quad 9B = 1,\quad 9A = 1,\quad B + D = 0,\quad A + C = 0 \\ \text{Hence } A = 1/9,\ B &= 1/9,\ D = -1/9,\ C = -1/9. \end{aligned}}$$

PROBLEM 2 sol^{n} continued

We found after some partial fractions,

$$\underline{Y}(s) = \frac{s+1-1/9}{s^2+9} + \frac{1}{9s^2} - \frac{1}{9}\left(\frac{1}{s} + \frac{1}{s^2} - \frac{s+1}{s^2+9}\right)e^{-s}$$

$$= \frac{s}{s^2+9} + \frac{8}{27}\left(\frac{3}{s^2+9}\right) + \frac{1}{9}\left(\frac{s}{s^2+9} + \frac{1}{3}\,\frac{3}{s^2+9} - \frac{1}{s} - \frac{1}{s^2}\right)e^{-s}$$

$$= \boxed{\cos 3t + \frac{8}{27}\sin 3t + \frac{1}{9}\left(\cos 3(t-1) + \frac{1}{3}\sin 3(t-1) - 1 - (t-1)\right)u(t-1)}$$

$$= \cos 3t + \frac{8}{27}\sin 3t + \frac{1}{9}\left(\cos(3t-3) + \frac{1}{3}\sin(3t-3) - t\right)u(t-1)$$