MATH 334: SPRING 2012:	QUIZ 6

Please put your work on this page. Box your answers. Thanks and enjoy. You have 15 minutes to complete this quiz. Choose two of 1a,1b,1c.

Problem 1a [3pts] Suppose $A = \begin{bmatrix} 6 & 8 \\ -3 & -4 \end{bmatrix}$. Solve $\frac{d\vec{x}}{dt} = A\vec{x}$.

Problem 1b [3pts] Suppose $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$. Solve $\frac{d\vec{x}}{dt} = A\vec{x}$.

1a) $\det\begin{pmatrix} 6-\lambda & 8 \\ -3 & -4-\lambda \end{pmatrix} = (\lambda-6)(\lambda+4)+24 = \lambda^2 - 2\lambda = \underbrace{\lambda(\lambda-2) = 0}_{\lambda_1 = 0,\ \lambda_2 = 2}$

$(A - 0I)\vec{u}_1 = \underbrace{\begin{bmatrix} 6 & 8 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}}_{-3u-4v=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-3u - 4v = 0 \Rightarrow v = \frac{-3}{4}u \Rightarrow \underline{\vec{u}_1 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}}$ (choose $u = 4$)

$(A - 2I)\vec{u}_2 = \underbrace{\begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}}_{4u+8v=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4u + 8v = 0 \Rightarrow v = -\frac{1}{2}u \Rightarrow \underline{\vec{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$ (choose $u = 2$)

Hence, $\boxed{\vec{x}(t) = c_1\begin{bmatrix} 4 \\ -3 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$

1b.) $\det\begin{pmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{pmatrix} = (4-\lambda)^2 = 0 \Rightarrow \underline{\lambda = 4 \text{ twice}}$.

$(A - 4I)\vec{u}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} u \text{ free} \\ v = 0 \end{matrix} \Rightarrow \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

Seek generalized e-vector $\vec{u}_2$ such that $(A-4I)\vec{u}_2 = \vec{u}_1$,

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} u \text{ free} \\ v = 1 \end{matrix} \Rightarrow \underline{\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$.

Notice $e^{At}\vec{u}_2 = e^{4t}(\vec{u}_2 + t(A-4I)\vec{u}_2 + \frac{t^2}{2}(A-4I)^2\vec{u}_2 + \cdots)$

Hence $\vec{x}_2(t) = e^{At}\vec{u}_2 = e^{4t}(\vec{u}_2 + t\vec{u}_1)$ "magic" formula.

Thus, $\boxed{\vec{x}(t) = c_1 e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{4t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}$

Problem 1c [3pts] Suppose $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$. Solve $\frac{d\vec{x}}{dt} = A\vec{x}$.

$$\det\begin{pmatrix} 1-\lambda & 3 \\ -3 & 1-\lambda \end{pmatrix} = (\lambda-1)^2 + 9 = 0 \hookrightarrow \lambda = 1 \pm 3i \quad \text{we } (+).$$

$$(A - (1+3i)I)\vec{u}_1 = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{-3iu + 3v = 0} \hookrightarrow v = iu$$

Let $u = 1$ to find $\vec{u}_1 = [1, i]^T$ thus,

$$\vec{x}(t) = e^{(1+3i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t(\cos 3t + i\sin 3t)\begin{bmatrix} 1 \\ i \end{bmatrix} \text{ is } \underline{\text{complex}} \text{ sol}^{\underline{n}}.$$

Select $\text{Re}(\vec{x}(t)) = e^t\begin{bmatrix} \cos 3t \\ -\sin 3t \end{bmatrix}$ and $\text{Im}(\vec{x}(t)) = e^t\begin{bmatrix} \sin 3t \\ \cos 3t \end{bmatrix}$ (real sol$^{\underline{n}}$s)

Therefore, $\boxed{\vec{x}(t) = c_1 e^t\begin{bmatrix} \cos 3t \\ -\sin 3t \end{bmatrix} + c_2 e^t\begin{bmatrix} \sin 3t \\ \cos 3t \end{bmatrix}}$

Problem 2 [2pts] Suppose $e^{tA} = \begin{bmatrix} e^t & e^{3t} \\ 2e^t & -e^{3t} \end{bmatrix}$. Find A.

The <u>idea</u> which I intended was $\frac{d}{dt}(e^{tA}) = Ae^{tA}$ (worked out in lecture) thus $A = \frac{d}{dt}(e^{tA})\Big|_{t=0}$.

$$\frac{d}{dt}\begin{bmatrix} e^t & e^{3t} \\ 2e^t & -e^{3t} \end{bmatrix} = \begin{bmatrix} e^t & 3e^{3t} \\ 2e^t & -e^{3t} \end{bmatrix}\Bigg|_{t=0} = \underline{\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}}.$$

However, I believe $\nexists\, A \in \mathbb{R}^{2\times 2}$ such that $e^{tA} = \begin{bmatrix} e^t & e^{3t} \\ 2e^t & -e^{3t} \end{bmatrix}$ hence my question is ill-posed.

Problem 3 [2pts] Suppose A is a 4×4 matrix with eigenvalues 1 and 3. Furthermore, suppose $\vec{u}_1, \vec{u}_2$ <u>is a chain for $\lambda = 1$</u> whereas $\vec{u}_3, \vec{u}_4$ are e-vectors for $\lambda_2 = 3$. (Solve $\vec{x}' = A\vec{x}$!)

$$(A - I)\vec{u}_1 = 0 \quad \& \quad (A - I)\vec{u}_2 = \vec{u}_1$$

$$(A - 3I)\vec{u}_3 = 0 \quad \& \quad (A - 3I)\vec{u}_4 = 0$$

$$e^{At}\vec{u}_2 = e^t\left(I + t(A-I) + \frac{t^2}{2}(A-I)^2 + \cdots\right)\vec{u}_2 = e^t(\vec{u}_2 + t\vec{u}_1).$$

The other three are e-vector sol$^{\underline{n}}$s, it follows,

$$\boxed{\vec{x}(t) = c_1 e^t\vec{u}_1 + c_2 e^t(\vec{u}_2 + t\vec{u}_1) + c_3 e^{3t}\vec{u}_3 + c_4 e^{3t}\vec{u}_4}$$