QUIZ 8 Sol^n

PROBLEM 1 Find complete power series sol^n of $\frac{dy}{dx} - 2xy = 0$ at zero.

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Thus

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0 \qquad \begin{cases} n-1 = j, \ j+1 = n \\ n+1 = j, \ j-1 = n \end{cases}$$

$$a_1 + \sum_{j=1}^{\infty}\Big[(j+1)a_{j+1} - 2a_{j-1}\Big]x^j = 0$$

Therefore, $a_1 = 0$ and $a_{j+1} = \left(\frac{2}{j+1}\right)a_{j-1}$ for $j = 1, 2, \dots$

I find it convenient to reformulate as $a_m = \frac{2}{m} a_{m-2}$ for $m = 2, 3, 4, \dots$

$$a_2 = \frac{2}{2}a_0, \quad a_4 = \frac{2}{4}a_2 = \frac{2\cdot 2}{4\cdot 2}a_0, \quad a_6 = \frac{2}{6}a_4 = \frac{2\cdot2\cdot2}{6\cdot4\cdot2}a_0$$

$$a_3 = \frac{2}{3}a_1 = 0, \quad a_5 = \frac{2}{5}a_3 = 0 \text{ etc.} \quad a_{2k+1} = 0$$

Continuing, we refine the a_{2k} formula,

$$a_8 = \frac{2\cdot2\cdot2\cdot2}{8\cdot6\cdot4\cdot2}a_0 = \frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}}{(4\cdot\cancel{2})(3\cdot\cancel{2})(2\cdot\cancel{2})(2\cdot1)}a_0 = \frac{a_0}{4!}$$

Suppose $a_{2k} = \frac{a_0}{(k)!}$ and consider inductively,

$$a_{2(k+1)} = \frac{2}{2(k+1)}a_{2k} = \frac{1}{k+1}\frac{a_0}{k!} = \frac{a_0}{(k+1)!}$$

Thus $a_{2k} = \frac{a_0}{k!}$ for all $k \geq 0$. Hence,

$$\boxed{y = \sum_{k=0}^{\infty} \frac{a_0}{k!}x^{2k} = a_0\sum_{k=0}^{\infty}\frac{(x^2)^k}{k!} = \underline{a_0 e^{x^2}}.}$$

PROBLEM 2 to QUIZ 8 Sol$^{\text{n}}$

Find 1$^{\text{st}}$ four nontrivial terms in Frobenius exp. of $3xy'' + (2-x)y' - y = 0$

Let $y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} =$

$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} =$

$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} =$

Note, the indicial eq$^{\text{n}}$ is found from lowest order.

$x^{r-2+1} = x^{r-1}$ terms

$3xy'' + 2y' - xy' - y = 0$

$\boxed{x^{r-1}}$ $3a_0 (r)(r-1) + 2a_0 r = 0$

$a_0 (3r(r-1) + 2r) = 0 \Rightarrow r(3r - 3 + 2) = 0$

$\Rightarrow r = 0, \; r = 1/3$

$\Rightarrow \underbrace{r_1 = 1/3}_{\text{custom to put this 1st.}}, \; r_2 = 0$

$y = x^{1/3}(a_0 + a_1 x + \cdots) = a_0 x^{1/3} + a_1 x^{4/3} + \cdots$

$y' = \frac{1}{3} a_0 x^{-2/3} + \frac{4}{3} a_1 x^{1/3} + \cdots$

$y'' = -\frac{2}{9} a_0 x^{-5/3} + \frac{4}{9} a_1 x^{-2/3} + \cdots$

$3xy'' + 2y' - xy' - y = 0$

$-\frac{2}{3} a_0 x^{-2/3} + \frac{4}{3} a_1 x^{1/3} + \frac{2}{3} a_0 x^{-2/3} + \frac{8}{3} a_1 x^{1/3} - \frac{1}{3} a_0 x^{1/3} - \frac{4}{3} a_1 x^{4/3} - a_0 x^{1/3} - a_1 x^{4/3} + \cdots = 0$

$x^{-2/3}$: $-\frac{2}{3} a_0 + \frac{2}{3} a_0 = 0.$

$x^{1/3}$: $\frac{4}{3} a_1 + \frac{8}{3} a_1 - \frac{1}{3} a_0 - a_0 = 0 \Rightarrow 12 a_1 = 4 a_0 \Rightarrow \underline{a_1 = \frac{1}{3} a_0}.$

$x^{4/3}$: no need, already have <u>two</u> here,

$$\underline{y_1(x) = a_0 \left(x^{1/3} + \frac{1}{3} x^{4/3} + \cdots \right)}.$$

Continuing Problem 2

$\underline{r_2 = 0}$ $\quad y = b_0 x^0 + b_1 x^1 + \cdots = b_0 + b_1 x + b_2 x^2 + \cdots$

$y' = b_1 + 2 b_2 x + \cdots$

$y'' = 2 b_2$

$3xy'' + 2y' - xy' - y = 0$

$6x b_2 + 2(b_1 + 2 b_2 x) - x(b_1) - b_0 - b_1 x = 0$ (upto x)

$\underline{\text{const}}$ $\quad 2b_1 - b_0 = 0 \Rightarrow \underline{b_1 = \frac{1}{2} b_0}$ done.

$\underline{x}$ $\quad 6b_2 + 4b_2 - b_1 - b_1 = 0$

$y_2 = b_0 \left(1 + \frac{1}{2} x + \cdots\right)$

Thus,

$$\boxed{y = a_0 \left(x^{1/3} + \frac{1}{3} x^{4/3} + \cdots\right) + b_0 \left(1 + \frac{1}{2} x + \cdots\right)}$$