Sol^n to Quiz 9

Problem 1 $y'' - 4y' + 13y = 0$ with $y(0) = 0$ & $y(\pi) = 0$

$$\lambda^2 - 4\lambda + 13 = (\lambda - 2)^2 + 9 = 0$$

$$y = c_1 e^{2t} \cos 3t + c_2 e^{2t} \sin 3t$$

$$y(0) = c_1 = 0$$
$$y(\pi) = c_2 e^{2\pi} \sin(6\pi) = 0 \Rightarrow c_2 \text{ free}$$

$$\boxed{y(t) = c_2 e^{2t} \sin 3t}$$

Problem 2 $y'' + Jy = 0$, $y(-\pi) = y(\pi)$, $y'(-\pi) = y'(\pi)$

$J = 0$ and $J < 0$ yield trivial sol^n's. Suppose $J = \beta^2$ for $\beta > 0$ then $y'' + \beta^2 y = 0$ hence $\lambda + \beta^2 = 0 \Rightarrow \lambda = \pm i\beta$.

$$y(t) = c_1 \cos \beta t + c_2 \sin \beta t$$
$$y'(t) = -c_1 \beta \sin \beta t + c_2 \beta \cos \beta t$$

$$y(\pi) = y(-\pi) \Rightarrow \cancel{c_1 \cos \beta\pi} + c_2 \sin \beta\pi = c_1 \cancel{\cos(-\beta\pi)} - c_2 \sin(\beta\pi)$$
$$2c_2 \sin \beta\pi = 0$$
$$\Rightarrow \beta\pi = n\pi, \quad \beta = n.$$

Likewise, $y'(-\pi) = y'(\pi)$ yields,

$$\cancel{c_2 \beta \cos(-\pi\beta)} - c_1 \beta \sin(-\beta\pi) = -c_1 \beta \sin \beta\pi + \cancel{c_2 \beta \cos \beta\pi}$$

Again $\beta = n$ fixes the sine terms above. We find,

$$\boxed{J = n^2 \text{ with } y_n(t) = c_n \cos(nt) + d_n \sin(nt)}$$

Problem 3 $u_t = u_{xx} - 4u$. Separate variables

Let $u(x,t) = X(x)\,T(t)$

$$u_t = X T'$$
$$u_{xx} = X'' T$$

Thus $u_t = u_{xx} - 4u \Rightarrow X T' = X'' T - 4 X T$

$$\frac{T'}{T} = \frac{X''}{X} - 4 = K$$

Hence, $\boxed{T' = KT \text{ and } X'' = (K+4)X}$

family of ODEs parametrized by e-value K.

Problem 4 Given sol^n to $u_{xx} = u_t$ with $u(0,t) = u(\pi,t) = 0$ for $t > 0$ has general sol^n $u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx)$. Suppose further that $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$ specifies $u_x(x,0) = f(x)$ (the initial velocity of u). Find the sol^n by specifying B_n

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \overbrace{\frac{\partial}{\partial x}\left[\sin(nx)\right]}^{n\cos(nx)}$$

$$u_x(x,0) = \sum_{n=1}^{\infty} n B_n \cos(nx) = f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$$

Thus $n B_n = \frac{1}{n^2} \Rightarrow \underline{B_n = \frac{1}{n^3}}$.

$$\therefore \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2 t} \sin(nx)}$$