MATH 334: SPRING 2012:	TEST 1

Please put your work on these pages. Box your answers. Thanks and enjoy. Problems 1,2 and 3 and 4 are basic problems which ought not involve a substitution (except, perhaps, to do an integral). However, Problem 5 is a Bernoulli equation as we discussed in lecture. Beyond that, it's up to your imagination.

Problem 1 [20pts] Find the explicit solution of $\frac{dy}{dx} = y$ through $(0, -1)$.

$$\frac{dy}{y} = dx \Rightarrow \ln|y| = x + c$$

$$|y| = e^{x+c} = e^x e^c$$

$$y = ke^x$$

$$y(0) = -1 \Rightarrow -1 = ke^0 \Rightarrow \boxed{y = -e^x}$$

Problem 2 [20pts] Find the explicit solution of $\frac{dy}{dx} = \sqrt{\frac{y+1}{x-1}}$ through $(2, 3)$.

$$\int \frac{dy}{\sqrt{y+1}} = \int \frac{dx}{\sqrt{x-1}} \Rightarrow 2\sqrt{y+1} = 2\sqrt{x-1} + C$$

$$\Rightarrow 2\sqrt{3+1} = 2\sqrt{2-1} + C$$

$$\Rightarrow 4 = 2 + C \quad \therefore \underline{C = 2}$$

Hence, $2\sqrt{y+1} = 2\sqrt{x-1} + 2$

$$\hookrightarrow y + 1 = (1 + \sqrt{x-1})^2$$

$$\boxed{y = -1 + (1 + \sqrt{x-1})^2}$$

Problem 3 [20pts] Find the explicit general solution of $\frac{dy}{dx} + \frac{2y}{x} = e^{x^3}$.

$$I = \exp\left(\int \frac{2dx}{x}\right) = \exp(2\ln|x|) = \exp(\ln|x|^2) = |x|^2 = x^2.$$

$$x^2 \frac{dy}{dx} + 2xy = x^2 e^{x^3}$$

$$\frac{d}{dx}(x^2 y) = x^2 e^{x^3}$$

$$x^2 y = \frac{1}{3} e^{x^3} + C$$

$$\boxed{y = \frac{1}{3x^2} e^{x^3} + C/x^2}$$

Problem 4 [30pts] Find the general implicit solution of $(2xy^2 + e^x)dx + (2x^2y - \sin(y))dy = 0$.

$$\frac{\partial F}{\partial x} = 2xy^2 + e^x \longrightarrow F(x,y) = x^2y^2 + e^x + C_1(y)$$

$$\frac{\partial F}{\partial y} = 2x^2y - \sin(y) \longrightarrow F(x,y) = x^2y^2 + \cos(y) + C_2(x)$$

Comparing we find $F(x,y) = x^2y^2 + e^x + \cos(y)$ will do.

Thus, the sol^n is

$$\boxed{x^2y^2 + e^x + \cos(y) = k}$$

Problem 5 [20pts] Find the explicit general solution of $\frac{dy}{dx} + \frac{3y}{x} = xy^2$.

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{3}{xy} = x \qquad \text{Let } z = \frac{1}{y} \therefore \frac{dz}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$$

$$-\frac{dz}{dx} + \frac{3}{x}z = x$$

$$\frac{dz}{dx} - \frac{3}{x}z = -x \qquad \hookrightarrow I = \exp\left(\int \frac{-3}{x}dx\right) = e^{\ln|x|^{-3}} = \frac{1}{x^3}$$

$$\frac{1}{x^3}\frac{dz}{dx} - \frac{3}{x^4}z = \frac{-1}{x^2}$$

$$\frac{d}{dx}\left[\frac{1}{x^3}z\right] = \frac{-1}{x^2}$$

$$\frac{1}{x^3}z = \frac{1}{x} + C \longrightarrow \frac{1}{y} = x^2 + Cx^3 \hookrightarrow \boxed{y = \frac{1}{Cx^3 + x^2}}$$

Problem 6 [20pts] Solve $\frac{dy}{dx} = (x + y + 3)^2$.

$$\text{Let } v = x + y + 3 \Rightarrow \frac{dv}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = v^2 \Rightarrow \frac{dv}{dx} = v^2 + 1$$

$$\int \frac{dv}{v^2+1} = \int dx \Rightarrow \tan^{-1}(v) = x + C$$

$$\Rightarrow \tan^{-1}(x + y + 3) = x + C$$

$$\boxed{y = -x - 3 + \tan(x + C)}$$

Problem 7 [20pts] Suppose a tank of salty water has $15kg$ of salt disolved in $1000L$ of water at time $t = 0$. Furthermore, assume pure water enters the tank at a rate of $10L/min$ and salty water drains out at a rate of $10L/min$. If $y(t)$ is the number of kg of salt at time t then **find** $y(t)$ **for** $t > 0$. We suppose that this tank is arranged such that the concentration of salt is constant throughout the liquid in this mixing tank.

$$\frac{dy}{dt} = -\left(\frac{10L}{min}\right)\left(\frac{y(t)}{1000L}\right) = \frac{-y}{100} \qquad \text{omitting } kg/min.$$

$$\frac{dy}{y} = -\frac{dt}{100} \longrightarrow \ln|y| = e^{\frac{-t}{100}} + C$$

$$\underline{y(t) = ke^{\frac{-t}{100}}}$$

$$y(0) = 15 = ke^0 \quad \therefore \quad k = 15.$$

$$\boxed{y(t) = 15e^{-t/100}}$$

Problem 8 [10pts] Let a, b be particular positive constants. Find the orthogonal trajectories to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

$$\frac{2x\,dx}{a^2} + \frac{2y\,dy}{b^2} = 0 \longrightarrow \frac{2y\,dy}{b^2} = \frac{-2x\,dx}{a^2}$$

$$\frac{dy}{dx} = \left(\frac{-b^2}{y}\right)\left(\frac{x}{a^2}\right) \xrightarrow{\text{O.T.}} \frac{dy}{dx} = \frac{a^2y}{b^2x}$$

$$\frac{dy}{a^2y} = \frac{dx}{b^2x}$$

$$\frac{1}{a^2}\ln|y| = \frac{1}{b^2}\ln|x| + C$$

$$\ln|y| = \ln|x|^{a^2/b^2} + C_2$$

$$|y| = \exp(\ln|x|^{a^2/b^2})\exp(C_2)$$

$$\boxed{y = k|x|^{a^2/b^2} = k|x|^{\left(\frac{a}{b}\right)^2}}$$

Problem 9 [20pts] Suppose a rocket car has an initial speed of v_o as it hurtles across a speedway in a remote desert. Suppose the driver opens a parachute which developes a retarding force proportional to the ~~cube~~ square of the velocity; $F_f = -kv^2$. Find the velocity as:

(a.) a function of time,

(b.) a function of position x taking x_o as the initial position

(a.)
$$m\frac{dv}{dt} = -kv^2 \Rightarrow \frac{dv}{-v^2} = \frac{k}{m}dt$$
$$\Rightarrow \frac{1}{v} = \frac{kt}{m} + C_1$$
$$\Rightarrow \frac{1}{v_o} = C_1$$
$$\therefore \boxed{v(t) = \frac{1}{\frac{1}{v_o} + \frac{kt}{m}} = \frac{v_o}{1 + \frac{v_o k t}{m}}}$$

(b.)
$$m\frac{dv}{dt} = -kv^2$$
$$m\frac{dx}{dt}\frac{dv}{dx} = -kv^2$$
$$mv\frac{dv}{dx} = -kv^2 \Rightarrow \frac{dv}{v} = \frac{-k}{m}dx$$
$$\Rightarrow \ln|v| = \frac{-kx}{m} + C_2$$
$$\Rightarrow \ln|v_o| = \frac{-kx_o}{m} + C_2$$
$$\ln|v| = \frac{-kx}{m} + \ln|v_o| + \frac{kx_o}{m}$$
$$\ln\left|\frac{v}{v_o}\right| = -\frac{k}{m}(x - x_o)$$
$$\frac{v}{v_o} = e^{-\frac{k}{m}(x-x_o)}$$
$$\boxed{v = v_o e^{\frac{-k}{m}(x-x_o)}}$$

Problem 10 [10pts] Suppose $\underbrace{\frac{dx}{dt}}_{}$ $= \underbrace{(x+y)^{42}}^{-N}$ and $\frac{dy}{dt} = \overbrace{cos(x+y) - \sqrt{x^2+y^2}}^{M}$. Give the first order ODE in Pfaffian form whose solutions are parametrized by the solutions of the given system of ODEs. **DO NOT ATTEMPT A SOLUTION OF THIS DEQn !**

$$M\,dx + N\,dy = 0 \quad \rightleftarrows \quad \left.\begin{aligned} \frac{dx}{dt} &= -N \\ \frac{dy}{dt} &= M \end{aligned}\right\} \text{one set-up possible}$$

... others exist.

$$\boxed{\left(\cos(x+y) - \sqrt{x^2+y^2}\right)dx - (x+y)^{42}\,dy = 0}$$

Problem 11 [10pts] Let $\star$ be the DEqn $y^4\cos(4x)dx + y^3 f(x)dy = 0$. Find all functions f such that $\star$ is an exact DEqn.

Need $\frac{\partial}{\partial y}\left(y^4\cos(4x)\right) = \frac{\partial}{\partial x}\left(y^3 f(x)\right)$

$$4y^3\cos(4x) = y^3\frac{\partial f}{\partial x} = y^3\frac{df}{dx}$$

Thus solve $\frac{df}{dx} = 4\cos(4x)$ integrate!

$$\boxed{f(x) = \sin(4x) + C}$$

Problem 12 [10pts] Suppose $\frac{dy}{dx} = \cos(x-y)$. Discuss the plotted solutions given below. In particular, comment on the uniqueness theorem and any apparent singular solutions.

$\frac{dy}{dx} = f(x,y)$ has unique sol^{n} at (x_o, y_o)
if we are given $\left.\frac{\partial f}{\partial y}\right|_{(x_o,y_o)} \neq 0$.

$$f(x,y) = \cos(x-y) \Rightarrow \frac{\partial f}{\partial y} = -\sin(x-y) = 0$$

for $x - y = n\pi$

$y = n\pi + x$

possible points of non-unique sol^{n}'s.
... potential singular sol^{n}'s.

turns out only even n are actually singular sol^{n}'s.

$n = 0$

$n = -2$