Please put your work on these pages. Box your answers. Thanks and enjoy. Remember, algebra matters.

Problem 1 [10pts] Find the general solution of $y'' + 3y' + 2y = 0$.

Problem 2 [5pts] Suppose a spring with mass $m = 1$, damping coefficient $\beta = 3$ and spring constant $k = 2$ has $y(0) = 0$ and $v(0) = 1$. Find the resulting equation of motion.

Problem 3 [10pts] Find the solution of $y'' + 9y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

Problem 4 [5pts] Suppose a spring with mass $m = 1$ is connected to a spring with spring constant $k = 9$ oscillates without friction. If $y(0) = 1$ and $v(0) = 0$ then find the resulting equation of motion.

Problem 5 [20pts] Solve $(D - 1)^2(D^2 - 2D + 37)[y] = 0$.

Problem 6 [10pts] Assume $x > 0$. Solve $x^2y'' - xy' - 3y = 0$.

Problem 7 [10pts] Assume $x > 0$. Solve $x^2y'' - xy' + 2y = 0$.

Problem 8 [20pts] Set-up the minimal particular solution for the differential equations below (do not find $A, B, \ldots$ etc..)

a. $y'' + 2y' + y = e^{2x} + 3x$

b. $(D^4 - 16)[y] = x\cos(2x)$

Problem 9 [40pts] Solve $y'' + y' = 2e^x + x^2$

Problem 10 [20pts] Solve $y'' + y = \csc(x)$

Problem 11 [20pts] Solve $y'' + y = \csc(x) + 2\sin(x)$. *Hint: use the previous problem!*

Problem 12 [10pts] Suppose $y_1(x) = e^x$, $y_2(x) = x$ and $y_3(x) = x^2$ are solutions to a third order linear ODE $L[y] = 0$. Answer, (Yes, No, Possible) and explain why you say what you say:

a. Is $\{y_1, y_2, y_3\}$ a fundamental solution set ?

b. Is $\{y_1, y_2, y_3\}$ a fundamental solution set of some constant coefficient third order ODE?

Problem 13 [20pts] Suppose T is an operator. We say $f_1 \neq 0$ is an **eigenfunction** of T with eigenvalue λ iff $(T - \lambda)[f_1] = 0$. If $f_2 \neq 0$ is a function with $(T - \lambda)^2[f_2] = 0$ then we call f_2 a generalized eigenfunction of order 2 with eigenvalue λ.

a. Suppose $z \neq 0$ satisfies $(T - \lambda)[z] = f_1$ where $(T - \lambda)[f_1] = 0$. Show that z is a generalized eigenfunction of order 2 with eigenvalue λ.

b. Show that $y_1(x) = x^3$ is an eigenfunction of $T = xD$.

c. Consider $(xD - 3)^2[y] = 0$. Find a generalized eigenfunction y_2 of order 2 with eigenvalue 3 for the xD operator by solving the chain condition given in part (a.) for the case $T = xD$.

d. use these results to solve the differential equation $(xD - 3)^2[y] = 0$

Problem 14 [10pts] Suppose $y_1(x) = 1, y_2(x) = x$ and $y_3(x) = x^2$ are solutions of $L[y] = 0$ on $(0, \infty)$. Furthermore, suppose L is a third-order linear differential operator. Solve $L[y] = x$.

Problem 15 [10pts] Derive Abel's formula for the $n = 2$ linear differential equation

$$ay'' + by' + cy = 0$$

You are given that y_1, y_2 are fundamental solutions and $a \neq 0$ on an interval I. Your goal is to find a formula for the Wronskian in terms of the arbitrary functions a and b. (curiously c does not appear in the formula)