Please put your work on these pages. Box your answers. Thanks and enjoy. Remember, algebra matters.

Problem 1 [10pts] Find the general solution of $y'' + 3y' + 2y = 0$.

$$\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x}}$$

Problem 2 [5pts] Suppose a spring with mass $m = 1$, damping coefficient $\beta = 3$ and spring constant $k = 2$ has $y(0) = 0$ and $v(0) = 1$. Find the resulting equation of motion.

$\ddot{y} + 3\dot{y} + 2y = 0$ Same problem.

$y(t) = c_1 e^{-t} + c_2 e^{-2t}$ $\quad y(0) = c_1 + c_2 = 0$

$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$ $\quad \dot{y}(0) = -c_1 - 2c_2 = 1$

$\Rightarrow -c_2 = 1 \quad \therefore \boxed{y = e^{-t} - e^{-2t}}$

Problem 3 [10pts] Find the solution of $y'' + 9y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

$y = c_1 \cos(3t) + c_2 \sin(3t)$

$y(0) = c_1 = 1$

$y'(0) = 3c_2 = 0 \quad \therefore \boxed{y(t) = \cos 3t}$

Problem 4 [5pts] Suppose a spring with mass $m = 1$ is connected to a spring with spring constant $k = 9$ oscillates without friction. If $y(0) = 1$ and $v(0) = 0$ then find the resulting equation of motion.

Same problem!

$$\boxed{y(t) = \cos 3t}$$

Problem 5 [20pts] Solve $(D - 1)^2(D^2 - 2D + 37)[y] = 0$.

$$(D-1)^2((D-1)^2 + 36)[y] = 0$$

$$\boxed{y = c_1 e^x + c_2 x e^x + c_2 e^x \cos 6x + c_3 e^x \sin 6x}$$

Problem 6 [10pts] Assume $x > 0$. Solve $x^2y'' - xy' - 3y = 0$.

Let $y = x^R$, $R(R-1) - R - 3 = 0$

$$R^2 - 2R - 3 = 0$$
$$(R-3)(R+1) = 0$$

Hence $R = 3, -1$ $\therefore$ $\boxed{y = C_1 x^3 + C_2/x}$

Problem 7 [10pts] Assume $x > 0$. Solve $x^2y'' - xy' + 2y = 0$.

$$R(R-1) - R + 2 = R^2 - 2R + 2$$
$$= (R-1)^2 + 1 = 0$$
$$R = 1 \pm i$$

$$\boxed{y = C_1 x \cos \ln(x) + C_2 x \sin \ln x}$$

Problem 8 [20pts] Set-up the minimal particular solution for the differential equations below (do not find $A, B, \ldots$ etc..)

a. $y'' + 2y' + y = e^{2x} + 3x$

$(D-2)\,D^2 = A$

$(D^2 + 2D + 1)[y] = e^{2x} + 3x$

$(D-2)D^2(D+1)^2[y] = 0 \longrightarrow y_p = Ae^{2x} + Bx + C$

b. $(D^4 - 16)[y] = x\cos(2x) \longrightarrow (D^2+4)^2$

$(D^2+4)^2(D^2+4)(D^2-4)[y] = 0$

$\longrightarrow \boxed{y_p = (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x}$

Problem 9 [40pts] Solve $y'' + y' = 2e^x + x^2 \longrightarrow \lambda^2 + \lambda = \lambda(\lambda+1) \longrightarrow y_h = C_1 + C_2 e^{-x}$

Let $y_p = Ae^x + Bx^3 + Cx^2 + Dx$

$$y_p' = Ae^x + 3Bx^2 + 2Cx + D$$
$$y_p'' = Ae^x + 6Bx + 2C$$

$$y_p'' + y_p' = 2Ae^x + (6B + 2C)x + (3B)x^2 + 2C + D = 2e^x + x^2$$

Thus $A = 1$, $6B + 2C = 0$, $2C + D = 0$

$3B = 1$

Hence $B = 1/3$, $C = \frac{-6B}{2} = -3B = -3(\frac{1}{3}) = -1$, $D = -2C = 2$

$$\boxed{y = C_1 + C_2 e^{-x} + e^x + \tfrac{1}{3}x^3 - x^2 + 2x}$$

$y_1 = \cos x\,,\quad y_2 = \sin x$

Problem 10 [20pts] Solve $y'' + y = \csc(x)$ $\quad y_1 y_2' - y_2 y_1' = 1$

$$V_2 = \int \frac{\cos(x)\csc(x)\,dx}{1} = \int \frac{\cos(x)\,dx}{\sin x} = \ln|\sin x|$$

$$V_1 = -\int \sin(x)\csc(x)\,dx = -\int dx = -x\,.$$

$$\boxed{y = c_1\cos x + c_2\sin x + \underbrace{\sin(x)\ln|\sin x| - x\cos x}_{y_{p_1}}}$$

f_1 note $y_{p_1}'' + y_{p_1} = \csc x$

Problem 11 [20pts] Solve $y'' + y = \csc(x) + 2\sin(x)$. *Hint: use the previous problem!*

Solve $y'' + y = 2\sin x$

$$(D^2+1)(D^2+1)[y] = (D^2+1)(2\sin x) = 0\,.$$

$$y_p = Ax\cos x + Bx\sin x$$

$$y_p' = A(\cos x - x\sin x) + B(\sin x + x\cos x)$$

$$y_p' = \cos x\,(A + Bx) + \sin x(-Ax + B)$$

$$y_p'' = -\sin x\,(A + Bx) + \cos x\,(B) + \cos x(-Ax + B) - A\sin x$$

$$y_p'' = \cos(x)\,[2B - Ax] + \sin x\,[-2A - Bx]$$

against y_p cancel

$$y_p'' + y_p = 2B\cos(x) - 2A\sin x = 2\sin x$$

$$2B = 0,\quad -2A = 2$$

Thus $y_{p_2} = -x\cos x$ solves $y'' + y = 2\sin x$.

Superposition yields, $\boxed{y = c_1\cos x + c_2\sin x + \sin x\ln|\sin x| - 2x\cos x}$

Problem 12 [10pts] Suppose $y_1(x) = e^x$, $y_2(x) = x$ and $y_3(x) = x^2$ are solutions to a third order linear ODE $L[y] = 0$. Answer, (Yes, No, Possible) and explain why you say what you say:

a. Is $\{y_1, y_2, y_3\}$ a fundamental solution set ? (YES.)

$$W[e^x, x, x^2; x] = \det \begin{bmatrix} e^x & x & x^2 \\ e^x & 1 & 2x \\ e^x & 0 & 2 \end{bmatrix} = e^x(2) - x(2e^x - 2xe^x) + x^2(e^x)$$
$$= e^x[2 - 2x + x^2]$$

$W(x) \neq 0$ for $x^2 - 2x + 2 = (x-1)^2 + 1 \neq 0$ $\therefore$ all $\mathbb{R}$!

b. Is $\{y_1, y_2, y_3\}$ a fundamental solution set of some constant coefficient third order ODE?

No. If $y_2 = x$, $y_3 = x^2$ then must be $\lambda = 0$ char. value $\Rightarrow$ $y_1 = 1 \neq e^x$!.

Problem 13 [20pts] Suppose T is an operator. We say $f_1 \neq 0$ is an **eigenfunction** of T with eigenvalue λ iff $(T - \lambda)[f_1] = 0$. If $f_2 \neq 0$ is a function with $(T - \lambda)^2[f_2] = 0$ then we call f_2 a generalized eigenfunction of order 2 with eigenvalue λ.

a. Suppose $z \neq 0$ satisfies $(T - \lambda)[z] = f_1$ where $(T - \lambda)[f_1] = 0$. Show that z is a generalized eigenfunction of order 2 with eigenvalue λ.

$$(T-\lambda)[z] = f_1 \Rightarrow (T-\lambda)[(T-\lambda)[z]] = (T-\lambda)[f_1] = 0.$$
$$\Rightarrow \underline{(T-\lambda)^2[z] = 0}.$$

Thus z is a generalized e-function with e-value λ.

b. Show that $y_1(x) = x^3$ is an eigenfunction of $T = xD - 3$.

$$T[x^3] = xD[x^3] = x(3x^2) = 3x^3$$

thus $(T-3)[x^3] = 0$ $\therefore$ $\boxed{y_1 = x^3 \text{ has } \lambda = 3 \text{ and is e-function of } T = xD}$

c. Consider $(xD - 3)^2[y] = 0$. Find a generalized eigenfunction y_2 of order 2 with eigenvalue 3 for the xD operator by solving the chain condition given in part (a.) for the case $T = xD$.

$$(xD - 3)[z] = x^3$$
$$x\frac{dz}{dx} - 3z = x^3$$
$$\underline{\frac{dz}{dx} - \frac{3}{x}z = x^2}$$

(assume $x > 0$) $I = \exp\left(\int \frac{-3\,dx}{x}\right) = e^{-3\ln|x|} = e^{\ln|x|^{-3}} = \frac{1}{x^3}$

$$\frac{1}{x^3}\frac{dz}{dx} - \frac{3}{x^4}z = \frac{1}{x}$$
$$\frac{d}{dz}\left[\frac{1}{x^3}z\right] = \frac{1}{x}$$
$$\frac{1}{x^3}z = \ln x + C_1$$
$$\boxed{z = \ln(x)x^3}$$

d. use these results to solve the differential equation $(xD - 3)^2[y] = 0$

$$\boxed{y = c_1x^3 + c_2\ln(x)x^3}$$

Problem 14 [10pts] Suppose $y_1(x) = 1, y_2(x) = x$ and $y_3(x) = x^2$ are solutions of $L[y] = 0$ on $(0, \infty)$. Furthermore, suppose L is a third-order linear differential operator. Solve $L[y] = x$.

$$W[1, x, x^2; x] = \det\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix} = \underline{2}.$$

$$\det(S_1) = \det\begin{bmatrix} 0 & x & x^2 \\ 0 & 1 & 2x \\ x & 0 & 2 \end{bmatrix} = x\det\begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix} = x(2x^2 - x^2) = \underline{x^3}.$$

$$\det(S_2) = \det\begin{bmatrix} 1 & 0 & x^2 \\ 0 & 0 & 2x \\ 0 & x & 2 \end{bmatrix} = -x\det\begin{bmatrix} 1 & x^2 \\ 0 & 2x \end{bmatrix} = \underline{-2x^2}.$$

$$\det(S_3) = \det\begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{bmatrix} = x.$$

We find $V_1 = \int \frac{x^3}{2}dx = \frac{1}{8}x^4$, $V_2 = \int \frac{-2x^2}{2}dx = \frac{-1}{3}x^3$, $V_3 = \int \frac{x}{2}dx = \frac{1}{4}x^2$

Thus, $y = c_1 + c_2 x + c_3 x^2 + \frac{1}{8}x^4 - \frac{1}{3}x^4 + \frac{1}{4}x^4$ $\qquad \frac{3-8+6}{24}$

$$\boxed{y = c_1 + c_2 x + c_3 x^2 + \frac{1}{24}x^4}$$

Problem 15 [10pts] Derive Abel's formula for the $n = 2$ linear differential equation

$$ay'' + by' + cy = 0 \longrightarrow y_j'' = \frac{-b}{a}y_j' - \frac{c}{a}y_j$$

You are given that y_1, y_2 are fundamental solutions and $a \neq 0$ on an interval I. Your goal is to find a formula for the Wronskian in terms of the arbitrary functions a and b. (curiously c does not appear in the formula)

$$W = y_1 y_2' - y_2 y_1'$$

$$W' = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$

$$= y_1 y_2'' - y_2 y_1''$$

$$= y_1\left[\frac{-b}{a}y_2' - \cancel{\frac{c}{a}y_2}\right] - y_2\left[\frac{-b}{a}y_1' - \cancel{\frac{c}{a}y_1}\right]$$

$$= \frac{-b}{a}(y_1 y_2' - y_2 y_1')$$

$$= \frac{-b}{a}W \quad \therefore \quad \frac{dW}{dx} = \frac{-b}{a}W$$

$$\int \frac{dW}{W} = \int \frac{-b}{a}dx$$

$$\ln|W| = \int \frac{-b}{a}dx$$

$$\boxed{W = c\exp\left(\int \frac{-b}{a}dx\right)}$$

Abel's Formula.