YOUR NAME HERE: ____________________

MATH 334	TEST III

This test is open book and you must show work for full credit. The only person you are allowed to get help with on this is me. You can consult static sources (books, non-response internet pages, wiki's etc...) and use Mathematica and/or Matlab etc... to solve integrations and/or produce plots. This is due on the reading day by 4:30pm. (remember the math suite closes at 5pm so getting it in on time is important, I will take off points for lateness here)

bonus: animate the solutions from either 9 or 10. Worth 20pts

Problem 1 [20pts] Find the first two terms in each of the Frobenius-type solutions of $4xy''+\frac{1}{2}y'+y=0$ at the regular singular point $x=0$. This means your general solution should have terms of up to (but not including) x^2.

$$y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \cdots$$

$$y' = ra_0x^{r-1} + a_1(r+1)x^r + a_2(r+2)x^{r+1} + \cdots$$

$$y'' = r(r-1)a_0x^{r-2} + a_1r(r+1)x^{r-1} + a_2(r+1)(r+2)x^r + \cdots$$

Substitute into the DEqt $4xy'' + \frac{1}{2}y' + y = 0$,

$$4r(r-1)a_0x^{r-1} + 4a_1r(r+1)x^r + 4a_2(r+1)(r+2)x^{r+1} + \cdots$$
$$+ \frac{1}{2}ra_0x^{r-1} + \frac{1}{2}a_1(r+1)x^r + \frac{1}{2}a_2(r+2)x^{r+1} + \cdots$$
$$+ a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \cdots = 0$$

Peel-off conditions, order-by-order,

I. $\underline{x^{r-1}}$ $\left(4r(r-1)+\frac{1}{2}r\right)a_0 = 0 \implies 4r^2 - \frac{7r}{2} = \underbrace{r\left(4r-\frac{7}{2}\right)}_{r_1=\frac{7}{8},\ r_2=0}$

II. $\underline{x^r}$ $\underbrace{4a_1r(r+1) + \frac{1}{2}a_1(r+1) + a_0 = 0}$

$$a_1 = \frac{-a_0}{(4r+1/2)(r+1)} \quad \text{III}$$

Apply $r_1 = 7/8$ to III, and then to $r_2 = 0$,

$$(r_1 = 7/8)\quad a_1 = \frac{-a_0}{\left(\frac{28}{8}+\frac{1}{2}\right)\left(\frac{7}{8}+1\right)} = \frac{-a_0}{\left(\frac{32}{8}\right)\left(\frac{15}{8}\right)} = \frac{-64a_0}{(15)(32)} = -\frac{2a_0}{15}.$$

$$(r_2=0)\quad a_1 = \frac{-a_0}{\left(\frac{1}{2}\right)(1)} = -2a_0$$

This suffices.

$$\boxed{y = c_1x^{7/8}\left(1-\frac{2}{15}x+\cdots\right) + c_2\left(1-2x+\cdots\right)}$$

Problem 2 [20pts] Solve $x' = 2x + y + 1$, $y' = 2y + z + 1$ and $z' = 2z + 1$ by whatever method you prefer.

① $z' = 2z + 1 \Rightarrow \underbrace{z' - 2z}_{\lambda - 2 = 0} = 1$

$\lambda - 2 = 0 \Rightarrow z_h(t) = c_1 e^{2t}$

Then $z_p = A \Rightarrow z_p' - 2z_p = 1 \Rightarrow -2A = 1 \therefore A = -1/2$

$$\boxed{z = c_1 e^{2t} - \frac{1}{2}}$$

② $y' = 2y + z + 1$

$y' - 2y = \underbrace{c_1 e^{2t} + \frac{1}{2}}_{\text{overlaps.}} \Rightarrow y = c_2 e^{2t} + y_p$

overlaps. $y_p = Ate^{2t} + B$

$y_p' - 2y_p = A(1 + \cancel{2t})e^{2t} - \cancel{2tAe^{2t}} - 2B = c_1 e^{2t} + \frac{1}{2}$

$\Rightarrow A = c_1$ and $-2B = 1/2 \therefore B = -1/4$

$$\boxed{y = c_2 e^{2t} + c_1 t e^{2t} - \frac{1}{4}}$$

③. $x' = 2x + y + 1$

$x' - 2x = c_2 e^{2t} + c_1 t e^{2t} + \frac{3}{4}$

$x(t) = c_3 e^{2t} + \underbrace{At^2 e^{2t} + B + Cte^{2t}}_{x_p}$

$x_p' - 2x_p = A(2te^{2t} + \cancel{2t^2 e^{2t}}) + C(1 + \cancel{2t})e^{2t} - \cancel{2At^2 e^{2t}} - 2B - \cancel{2Cte^{2t}}$
$= c_2 e^{2t} + c_1 t e^{2t} + \frac{3}{4}$

te^{2t} | $2A = c_1 \Rightarrow A = c_1/2$
e^{2t} | $C = c_2 \Rightarrow C = c_2$
1 | $-2B = 3/4 \Rightarrow B = -3/8$

$$\boxed{x = c_3 e^{2t} + c_2 t e^{2t} + \frac{1}{2} c_1 t^2 e^{2t} - \frac{3}{8}}$$

$$\boxed{\vec{X}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3/8 \\ -1/4 \\ -1/2 \end{bmatrix}}$$

Problem 3 [20pts] Solve $x'' + 10x - 4y = 0$ and $-4x + y'' + 4y = 0$ subject the initial conditions $x(0) = 0, x'(0) = 1$ and $y(0) = 0$ and $y'(0) = -1$. *Hint: you could use Laplace transforms here, you'd get a system in X and Y to unravel... or you could use the method of Chapter 5 we dicussed right after Test 2, but then you have to fit initial conditions... your choice*

$$s^2 X - 1 + 10X - 4Y = 0$$

$$-4X + s^2 Y + 1 + 4Y = 0$$

Thus,

$$(s^2+10)X - 4Y = 1 \Rightarrow \underbrace{Y = \left(\frac{s^2+10}{4}\right)X - \frac{1}{4}}$$

$$-4X + (s^2+4)Y = -1$$

$$-4X + (s^2+4)\left(\left(\frac{s^2+10}{4}\right)X - \frac{1}{4}\right) = -1$$

$$-16X + (s^2+4)\left[(s^2+10)X - 1\right] = -4$$

$$-16X + (s^2+4)(s^2+10)X - (s^2+4) = -4$$

$$(s^4 + 14s^2 + 24)X = s^2$$

$$X = \frac{s^2}{(s^2+2)(s^2+12)} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2+12}$$

$$s^2 = (As+B)(s^2+12) + (Cs+D)(s^2+2)$$

$$s^2 = s^3(A+C) + s^2(B+D) + s(12A+2C) + 12B + 2D$$

$s^3\rfloor$ $0 = A + C \Rightarrow A = -C \Rightarrow 10A = 0 \therefore \underline{A = C = 0}$.

$s^2\rfloor$ $1 = B + D$

$s\rfloor$ $0 = 12A + 2C$

$\text{Const}\rfloor$ $0 = 12B + 2D \Rightarrow D = -6B \Rightarrow 1 = B - 6B = -5B \quad \therefore B = -1/5$ $\;\&\; D = 6/5$

Thus $X(s) = \frac{-1}{5\sqrt{2}}\left(\frac{\sqrt{2}}{s^2+2}\right) + \frac{6}{5\sqrt{12}}\left(\frac{\sqrt{12}}{s^2+12}\right)$

Therefore, $\boxed{x(t) = \frac{-1}{5\sqrt{2}}\sin(\sqrt{2}\,t) + \frac{6}{5\sqrt{12}}\sin(\sqrt{12}\,t)}$ continued

&

$\boxed{y(t) = \frac{-2}{5\sqrt{2}}\sin(\sqrt{2}\,t) - \frac{3}{5\sqrt{12}}\sin(\sqrt{12}\,t)}$

See next page for why.

Problem 3 Continued

$$Y = \left(\frac{s^2+10}{4}\right)\left(\frac{s^2}{(s^2+2)(s^2+12)}\right) - \frac{1}{4}$$

$$= \frac{1}{4}\left[\frac{(s^2+10)s^2 - (s^2+2)(s^2+12)}{(s^2+2)(s^2+12)}\right]$$

$$= \frac{1}{4}\left[\frac{\cancel{s^4} + 10s^2 - \cancel{s^4} - 14s^2 - 24}{(s^2+2)(s^2+12)}\right]$$

$$= \frac{1}{4}\left[\frac{-4s^2 - 24}{(s^2+2)(s^2+12)}\right]$$

$$= \frac{-s^2 - 6}{(s^2+2)(s^2+12)} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2+12}$$

$$-s^2 - 6 = (As+B)(s^2+12) + (Cs+D)(s^2+2)$$

$$-s^2 - 6 = s^3(A+C) + s^2(B+D) + s(12A+2C) + 12B + 2D$$

$0 = A + C$ $\longrightarrow$ $\underline{A = C = 0}$.

$-1 = B + D$

$0 = 12A + 2C$

$-6 = 12B + 2D$ $\longrightarrow$ $-6 = 12B + 2(-1-B)$

$\Rightarrow -4 = 10B$

$\Rightarrow B = \frac{-2}{5}$ and $D = -1 - B = -1 + \frac{2}{5} = \frac{-3}{5}$.

Thus,

$$Y(s) = \frac{-2}{5\sqrt{2}}\left(\frac{\sqrt{2}}{s^2+2}\right) - \frac{3}{5\sqrt{12}}\left(\frac{\sqrt{12}}{s^2+12}\right)$$

Therefore,

$$\boxed{y(t) = \frac{-2}{5\sqrt{2}}\sin(\sqrt{2}\,t) - \frac{3}{5\sqrt{12}}\sin(\sqrt{12}\,t)}$$

Problem 4 [20pts] Solve $x' = x - 2y$ and $y' = 5x - y$ by the eigenvector method.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & -2 \\ 5 & -1-\lambda \end{bmatrix}$$
$$= (\lambda-1)(\lambda+1) + 10$$
$$= \lambda^2 + 9 \implies \underline{\lambda = \pm 3i}$$

<u>Find the e-vector with $\lambda = 3i$</u>

$$(A - 3iI)\vec{u} = \begin{bmatrix} 1-3i & -2 \\ 5 & -1-3i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-3i)u - 2v = 0 \implies v = \left(\frac{1-3i}{2}\right)u$$

Let $u = 2$ hence $v = 1 - 3i$, $\implies \underline{\vec{u} = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix}}$

$$\vec{z}(t) = e^{3it}\begin{bmatrix} 2 \\ 1-3i \end{bmatrix} = (\cos(3t) + i\sin(3t))\begin{bmatrix} 2 \\ 1-3i \end{bmatrix}$$
$$= \begin{bmatrix} 2\cos 3t \\ \cos 3t + 3\sin 3t \end{bmatrix} + i\begin{bmatrix} 2\sin 3t \\ \sin 3t - 3\cos 3t \end{bmatrix}$$

$$\implies \boxed{\vec{X}(t) = c_1\begin{bmatrix} 2\cos 3t \\ \cos 3t + 3\sin 3t \end{bmatrix} + c_2\begin{bmatrix} 2\sin 3t \\ \sin 3t - 3\cos 3t \end{bmatrix}}$$

Problem 5 [20pts] Solve $y''+4y'+13y=\delta(t-\pi)+\delta(t-3\pi)$ with initial conditions $y(0)=1, y'(0)=0$.

$$s^2 Y - s + 4s Y - 4 + 13 Y = e^{-\pi s} + e^{-3\pi s}$$

$$(s^2+4s+13) Y = s+4 + e^{-\pi s} + e^{-3\pi s}$$

$$Y(s) = \frac{s+2+\frac{2}{3}(3)}{(s+2)^2+9} + \frac{1}{3}\left[\frac{3}{(s+2)^2+9}\right]\left(e^{-\pi s}+e^{-3\pi s}\right)$$

$$\boxed{\begin{aligned} y(t) = {} & e^{-2t}\left(\cos 3t + \tfrac{2}{3}\sin 3t\right) + \\ & + \tfrac{1}{3}\sin(3(t-\pi))\, e^{-2(t-\pi)}\, u(t-\pi) \\ & + \tfrac{1}{3}\sin(3(t-3\pi))\, e^{-2(t-3\pi)}\, u(t-3\pi) \end{aligned}}$$

$$\sin[3(t-\pi)] = \sin(3t-3\pi) = \sin 3t \cos(-3\pi) - \sin 3\pi \cos 3t = -\sin 3t.$$

$$\sin[3(t-3\pi)] = \sin(3t-9\pi) = \sin 3t \cos(-9\pi) - \sin 9\pi \cos 3t = -\sin 3t.$$

Thus,

$$\boxed{y(t) = e^{-2t}\left(\cos 3t + \tfrac{2}{3}\sin 3t\right) - \frac{e^{-2(t-\pi)}}{3}\sin(3t)\, u(t-\pi) - \frac{e^{-2(t-3\pi)}}{3}\sin(3t)\, u(t-3\pi)}$$

Problem 6 [20pts] Let $f(t) = \begin{cases} t & 0 < t < 1 \\ \sin(t) & t > 1 \end{cases}$. Suppose $y(0) = y'(0) = 0$. Solve $y'' = f(t)$.

$$f(t) = t\,[u(t) - u(t-1)] + \sin t\,[u(t-1)]$$

$$f(t) = [\sin t - t]\,u(t-1) + t\,u(t)$$

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{\sin(t+1) - (t+1)\}(s)\,e^{-s} + \frac{1}{s^2}$$

$$= \mathcal{L}\{\sin(1)\cos t + \cos(1)\sin t - t - 1\}(s)\,e^{-s} + \frac{1}{s^2}$$

$$= \left[\frac{\sin(1)\,s + \cos(1)}{s^2+1} - \frac{1}{s^2} - \frac{1}{s}\right]e^{-s} + \frac{1}{s^2}$$

$$\mathcal{L}\{y''\} = \mathcal{L}\{f\}$$

$$s^2 Y = \left[\frac{\sin(1)\,s + \cos(1)}{s^2+1} - \frac{1}{s^2} - \frac{1}{s}\right]e^{-s} + \frac{1}{s^2}$$

$$Y(s) = \left[\frac{\sin(1)\,s + \cos(1)}{s^2(s^2+1)} - \frac{1}{s^4} - \frac{1}{s^3}\right]e^{-s} + \frac{1}{s^4}$$

$$\Rightarrow Y(s) = \left(\left[\sin(1)\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \cos(1)\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] - \frac{1}{s^4} - \frac{1}{s^3}\right)e^{-s} + \frac{1}{s^4}$$

$$\boxed{y(t) = \left(\sin(1)\,[1 - \cos(t-1)] + \cos(1)\,[t-1-\sin(t-1)] - \frac{1}{6}(t-1)^3 - \frac{1}{2}(t-1)^2\right)u(t-1) + \frac{1}{6}t^3}$$

Problem 7 [20pts] Find the inverse Laplace transform of

$$F(s) = \frac{3se^{-2s}}{s^3 + 8s^2 + 20s}$$

<u>Remark</u>: Sorry I should have asked $(3s+1)e^{-2s}$ ... this one is too easy ☺

$$F(s) = \frac{3e^{-2s}}{s^2 + 8s + 20}$$

$$= \frac{3}{2}\left[\frac{2}{(s+4)^2 + 4}\right]e^{-2s}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$$

$$= \boxed{\frac{3}{2}e^{-4(t-2)}\sin(2(t-2))\,u(t-2)}$$

Problem 8 [20pts] Suppose $B(t) = \begin{bmatrix} 1+t & 4t \\ 2t & 1-2t \end{bmatrix}$.

1. calculate $B'(t)$
2. calculate $B'(0)$
3. let A be a square matrix, show that $\frac{d}{dt}\left[e^{At}\right]\Big|_{t=0} = A$.
4. show that if $Av = \lambda v$ then $e^{At}\vec{v} = e^{\lambda t}\vec{v}$
5. is it possible that $B(t) = e^{tA}$ for some matrix A?

1.) $B'(t) = \begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix} = B'(0)$ (2.)

3.)
$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{n t^{n-1}}{n!} A^n$$
$$= A \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} A^{n-1}$$
$$= A \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j$$
$$= Ae^{At} \therefore \frac{d}{dt}\left(e^{At}\right)\Big|_{t=0} = Ae^0 = A.$$

4.) Notice that
$$A\vec{v} = \lambda\vec{v} \Rightarrow A^2\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}$$
$$\Rightarrow A^3\vec{v} = \lambda^2 A\vec{v} = \lambda^3\vec{v} \cdots \Rightarrow A^k\vec{v} = \lambda^k\vec{v}$$
$$e^{At}\vec{v} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \vec{v} = \left(\sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!}\right)\vec{v} = e^{\lambda t}\vec{v}.$$

5.) If $B(t) = e^{tA}$ then by 2.
$$A = \begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix}$$
$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 4 \\ 2 & -2-\lambda \end{bmatrix} = (\lambda-1)(\lambda+2) - 8 = \lambda^2 + \lambda - 10$$
$$\Rightarrow \left(\lambda + \tfrac{1}{2}\right)^2 - \tfrac{41}{4} = 0$$
Thus $\lambda = \frac{-1 \pm \sqrt{41}}{2}$. We calculate the eigenvector $(A - \lambda I)\vec{v} = \begin{bmatrix} 1-\lambda & 4 \\ 2 & -2-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2u = (2+\lambda)v$

hence $u = \left(\frac{2+\lambda}{2}\right)v \Rightarrow u = 2+\lambda$ & $v = 2$ where $\lambda = \frac{-1+\sqrt{41}}{2}$.

Consider $B(t)\vec{v} = \begin{bmatrix} 1+t & 4t \\ 2t & 1-2t \end{bmatrix}\begin{bmatrix} 2+\lambda \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} (1+t)(2+\lambda) + 8t \\ 4t + 2t\lambda + 2 - 4t \end{bmatrix}}_{\text{clearly not an exponential!}} \neq e^{\lambda t}\vec{v}$.

you can work together on Problems 9 and 10, but NOT the other problems! This you may recognize as Problem 69 and 71 from Problem Set IV)

Problem 9 [20pts] Solve the heat equation $u_t = u_{xx}$ on $0 < x < \pi$ for $t > 0$ subject to the boundary conditions $u_x(0,t) = u_t(\pi,t) = 0$ for $t > 0$ and the initial condition $u(x,0) = e^x$ for $0 < x < \pi$.

Problem 10 [20pts] Solve $u_{tt} = u_{xx}$ for $0 < x < 1$ and $t > 0$ subject the boundary conditions $u(0,t) = u(1,t) = 0$ for $t > 0$ given the initial conditions $u(x,0) = x(1-x)$ and $u_t(x,0) = \sin(5\pi x) + \sin(10\pi x)$.

you may use the general solution given on PH-146-147, the solution of 10.6 number 1 of Nagel Saff and Snider. You just need to understand the solution well enough to slightly modify it. You can just outline the calculation, all the details need not be repeated. Do box your answer as always!

$T = \pi$

Problem 68 Find the Fourier expansion of $f(x) = e^x$ on $0 < x < \pi$

Suppose $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = e^x$

Then we can calculate the Fourier coeff. $a_0, a_1, \ldots$ by:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} e^x\,dx = \frac{2}{\pi}(e^{\pi} - 1)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos(nx)\,dx$$

$\int e^x \cos(nx)\,dx = e^x(A\cos(nx) + B\sin(nx))$ now find A, B

$$e^x\cos(nx) = e^x\big((A + nB)\cos(nx) + (B - nA)\sin(nx)\big)$$

thus $A + nB = 1$ and $B - nA = 0 \Rightarrow \underline{B = nA}$.

$A + n^2 A = 1 \Rightarrow A = \frac{1}{n^2+1}$ & $B = \frac{n}{n^2+1}$

$$\int e^x \cos(nx)\,dx = e^x\left(\frac{1}{n^2+1}\cos(nx) + \frac{n}{n^2+1}\sin(nx)\right) + C$$

$$a_n = \frac{2}{\pi}\cdot\frac{e^x\cos(nx)}{n^2+1}\Big|_0^{\pi}$$ (note that $\sin n\pi = \sin 0 = 0$)

$$a_n = \frac{2}{\pi}\left(\frac{e^{\pi}\cos(n\pi) - 1}{n^2+1}\right), \quad \cos(n\pi) = (-1)^n$$

$$\boxed{f(x) \sim \underbrace{\frac{e^{\pi} - 1}{\pi}}_{a_0/2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n e^{\pi} - 1\right]}{\pi(n^2+1)}\cos(nx)}$$

<u>Remark</u>: I meant to ask you to plot these for $n = 0, 1, 2, 3, 4, 5$.

Problem 69 Solve the heat equation $u_t = u_{xx}$ on $0 < x < \pi$ for $t > 0$ subject to the boundary conditions $u_x(0,t) = \cancel{u_t}(\pi,t) = 0$ for $t > 0$ and the initial condition $u(x,0) = e^x$ for $0 < x < \pi$.

x (typo)

$$u(x,t) = X(x)T(t) \Rightarrow XT' = X''T \Rightarrow \frac{T'}{T} = \frac{X''}{X} = k$$

Suppose $k = -\beta^2$ for $\beta > 0$ then $X'' + \beta^2 X = 0$ hence $X(x) = A\cos\beta x + B\sin\beta x$. Note $u_x(0,t) = u_x(\pi,t) = 0$ yields $X'(0) = X'(\pi)$. Observe $X'(x) = -A\beta\sin\beta x + B\beta\cos\beta x$ thus

$$X'(0) = B\beta = 0 \Rightarrow \underline{B = 0}.$$

$$X'(\pi) = -A\beta\sin\beta\pi = 0 \Rightarrow \beta\pi = n\pi \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow \underline{\beta = n \text{ for } n \in \mathbb{N}}.$$

It follows that $T' = -n^2 T \Rightarrow T(t) = \exp(-n^2 t)$

The general sol^n to the BVP is $\underline{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t}\cos(nx) + A_0}$

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos(nx) = \underbrace{\frac{e^{\pi}-1}{\pi}}_{A_0} + \sum_{n=1}^{\infty} \underbrace{\frac{2[(-1)^n e^{\pi} - 1]}{\pi(n^2+1)}}_{A_n}\cos(nx)$$

also satisfies BVP.

Thus,

$$\boxed{u(x,t) = \frac{e^{\pi}-1}{\pi} + \sum_{n=1}^{\infty} \frac{2[(-1)^n e^{\pi} - 1]}{\pi(n^2+1)} e^{-n^2 t}\cos(nx)}$$

differs from §10.6 #1.

Problem 71 Solve $u_{tt} = u_{xx}$ for $0 < x < 1$ and $t > 0$ subject the boundary conditions $u(0,t) = u(1,t) = 0$ for $t > 0$ given the initial conditions $u(x,0) = x(1-x)$ and $u_t(x,0) = \sin(5\pi x) + \sin(10\pi x)$.
you may use the general solution given on PH-146-147, the solution of 10.6 number 1 of Nagel Saff and Snider. You just need to understand the solution well enough to slightly modify it. You can just outline the calculation, all the details need not be repeated. Do box your answer as always!

$$(\text{PH } 146-147) \Longrightarrow u(x,t) = \sum_{n=1}^{\infty}\Big(A_n\cos(n\pi t)\sin(n\pi x) + B_n\sin(n\pi t)\sin(n\pi x)\Big)$$

$$\text{But, } u(x,0) = \sum_{n=1}^{\infty} A_n\sin(n\pi x) = x - x^2 = \underbrace{\sum_{k=1}^{\infty}\frac{8}{(2k-1)^3\pi^3}\sin((2k-1)\pi x)}_{\text{By } \underline{\text{PH147}}}.$$

$$\text{Thus } A_{2k} = 0 \text{ whereas } A_{2k-1} = \frac{8}{(2k-1)^3\pi^3}.$$

On the other hand,

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\Big(-n\pi A_n\sin(n\pi t)\sin n\pi x + n\pi B_n\cos(n\pi t)\sin n\pi x\Big)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} n\pi B_n\sin(n\pi x) = \sin(5\pi x) + \sin(10\pi x)$$

$$\Longrightarrow 5\pi B_5 = 1 \ \& \ 10\pi B_{10} = 1 \ \& \ B_n = 0 \text{ otherwise.}$$

Thus,

$$\boxed{u(x,t) = \frac{1}{5\pi}\sin(5\pi t)\sin(5\pi x) + \frac{1}{10\pi}\sin(10\pi t)\sin(10\pi x) + \sum_{k=1}^{\infty}\frac{8}{(2k-1)^3\pi^3}\sin[(2k-1)\pi x]\cos((2k-1)\pi t)}$$