Make sure you name is on each page and the assignment is stapled. Thanks and enjoy.

Problem 1 Find power series expansion for $f(x) = x^2 \sin(x^3)$ centereed at $x_o = 0$. Do not do this by Taylor's Theorem directly!

Problem 2 exercise 1 of page 438 in text.

Problem 3 exercise 16 of page 439 in text.

Problem 4 exercise 23 of page 440 in text.

Problem 5 exercise 24 of page 440 in text.

Problem 6 exercise 32 of page 440 in text.

Problem 7 exercise 13 of page 450 in text.

Problem 8 exercise 22 of page 450 in text.

Problem 9 exercise 25 of page 450 in text.

Problem 10 exercise 17 of page 456 in text.

Problem 11 Find the power series expansion for $f(x) = x\sin(x)$ centered at $x_o = 3$.

Problem 12 Find the power series expansion of $f(x) = \frac{x^2}{2+x}$ centered at $x_o = -1$.

Problem 13 Define $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ and likewise for e^y. Prove $e^x e^y = e^{x+y}$ by multiplying the series for e^x and e^y. Use the Cauchy product defined on pg. 434 of your text to multiply the series.

Problem 14 Suppose $\sum_{k=0}^{\infty}(a_{2k}x^{2k} + b_{2k+1}x^{2k+1} = e^x + \cos(x+2)$. Find explicit formulas for a_{2k} and b_{2k+1} via Σ-notation algebra.

Problem 15 Find a power series solution to the integrals below:

(a.) $\int \frac{x^3+x^6}{1-x^3}\,dx$

(b.) $\int x^8 e^{x^3+2}\,dx$

Problem 16 Calculate the 42^{nd}-derivative of $x^2\cos(x)$ at $x = 1$. (use power series techniques)

Problem 17 Find the complete power series solution of $y'' + x^2y' + 2xy = 0$ about the ordinary point $x = 0$. Your answer should include nice formulas for arbitrary coefficients in each of the fundamental solutions. You need to both set-up and solve the reccurrence relations as best you can.

Problem 18 (Ritger & Rose 7-2 problem 7 part c) Find the first four nonzero terms in the power series solution about zero for the initial value problem $(x+2)y'' + 3y = 0$ with $y(0) = 0$ and $y'(0) = 1$.

Problem 19 (Ritger & Rose 7-2 problem 7 part d) Find the first four nonzero terms in the power series solution about zero for the initial value problem $y'' + \sin(x)y' + (x-1)y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

Problem 20 Construct a differential equation with $y_1(x) = \frac{\sin(x)}{x}$ for $x \neq 0$ and $y_1(0) = 1$, $y_2(x) = x$ as its fundamental solution set. To accomplish this task do two tasks:

(a.) Argue from appropriate facts from the theory of determinants that $L[y] = \det \begin{bmatrix} y & y' & y'' \\ y_1 & y_1' & y_1'' \\ y_2 & y_2' & y_2'' \end{bmatrix}$ is a linear ODE with solutions y_1 and y_2.

(b.) calculate $L[y]$ explicitly as a linear ODE of the form $py'' + qy' + ry = 0$ where p, q, r are perhaps given as Taylor expansions about zero. (just find the first few terms in the taylor expansions of the coefficient functions p, q, r)

P1 $f(x) = x^2 \sin(x^3)$

$$= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^3)^{2n+1}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{6n+5}}$$

P2 #1 from pg. 438

$$\sum_{n=0}^{\infty} \underbrace{\frac{2^{-n}}{n+1} (x-1)^n}_{a_n}$$ find IOC (Interval of Convergence)

Ratio Test $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{-n-1}(x-1)^{n+1}}{n+2} \cdot \frac{n+1}{2^{-n}(x-1)^n}\right| = \frac{n+1}{2(n+2)}|x-1|$

Hence, $\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n+2}\right)\frac{1}{2}|x-1| \longrightarrow \frac{1}{2}|x-1| < 1 \Rightarrow |x-1| < 2$.

$x \in (1-2, 1+2)$

Test Endpts of $(-1, 3)$

$x=-1$ $\sum_{n=0}^{\infty} \frac{2^{-n}(-2)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$: converges by Alt. series test.

$x=3$ $\sum_{n=0}^{\infty} \frac{2^{-n}2^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$: diverges by $p=1$ series (harmonic series)

Thus, $\boxed{IOC = [-1, 3)}$

P3 #16 of pg. 439

$$\begin{array}{r l}
 & 1 - \frac{1}{2}x + \frac{1}{4}x^2 + \cdots \\
1 + x + \frac{1}{2}x^2 + \cdots & \overline{)\,1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots} \\
 & \underline{1 + x + \frac{1}{2}x^2} \\
 & -\frac{1}{2}x - \frac{1}{4}x^2 \\
 & \underline{-\frac{1}{2}x - \frac{1}{2}x^2} \\
 & \frac{1}{4}x^2 \\
 & \underline{\frac{1}{4}x^2} \\
 & \cdots
\end{array}$$

Hence, $$\frac{1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots}{1 + x + \frac{1}{2}x^2 + \cdots} = \boxed{1 - \frac{1}{2}x + \frac{1}{4}x^2 + \cdots}$$

P4 #23 p. 440

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \boxed{\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k}$$

$k = n-1$
$n = k+1$
$n = 1 \Rightarrow k = 0$

P5 #24 p. 440

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n+2} = \boxed{\sum_{k=4}^{\infty} (k-2)(k-3) a_{k-2} x^k}$$

Setting $k = n+2$
$n = k-2$, $n-1 = k-3$
$n = 2 \Rightarrow k = 4$.

P6 #32 p. 440 find Taylor series centered about $x_0 = 0$,

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f^{(3)}(x) = \frac{(-2)(-1)}{(1+x)^3}$

$f^{(4)}(x) = \frac{(-3)(-2)(-1)}{(1+x)^4}$

$\vdots$

$f^{(n)}(x) = \frac{(n-1)!\,(-1)^{n+1}}{(1+x)^n}$

Hence, $f^{(n)}(0) = (n-1)!\,(-1)^{n+1}$. If $n \neq 0$,

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}(n-1)!}{n\,(n-1)!} = \frac{(-1)^{n+1}}{n}$$

Thus, as $f(0) = \ln(1) = 0$,

$$\boxed{\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n}$$

P7 # 13 pg. 450 |

$z'' - x^2 z = 0$ Find 1^{st} 4 terms in gen. sol$^{\underline{t}}$ at $x=0$

$$z = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$z'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots$$

$$z'' - x^2 z = (2a_2 + 6a_3 x + 12a_4 x^2 + \cdots) - x^2(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots$$

$$- a_0 x^2 - a_1 x^3 - a_2 x^4 - \cdots$$

$$= 2a_2 + 6a_3 x + (12a_4 - a_0) x^2 + (20a_5 - a_1) x^3$$

$$+ (30a_6 - a_2) x^4 + \cdots$$

Hence, equating coeff.,

$$a_2 = 0, \quad a_3 = 0, \quad \underbrace{12a_4 - a_0 = 0}_{a_4 = \frac{a_0}{12}}, \quad \underbrace{20a_5 - a_1 = 0}_{a_5 = \frac{a_1}{20}}, \ldots$$

Thus, $\boxed{z = a_0\left(1 + \frac{1}{12}x^4 + \cdots\right) + a_1\left(x + \frac{1}{20}x^5 + \cdots\right)}$

P8 # 22 pg. 450

$$\boxed{y'' - xy = 0}$$ (find general sol^{n}) (about $x_0 = 0$)

Suppose $y = \sum_{n=0}^{\infty} a_n x^n$ then $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Hence, $y'' - xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}$

$n-2 = j$
$n = j+2$
OR
$n+1 = j$
$n = j-1$

$$= a_2 + \sum_{j=1}^{\infty} \left((j+2)(j+1) a_{j+2} - a_{j-1} \right) x^j$$

$$= a_2 + \sum_{j=1}^{\infty} \left[(j+1)(j+2) a_{j+2} - a_{j-1} \right] x^j$$

Hence, $a_2 = 0$ and, $a_{j+2} = \dfrac{a_{j-1}}{(j+2)(j+1)}$ for $j = 1, 2, \ldots$

Instead, $a_n = \dfrac{a_{n-3}}{n(n-1)}$ for $n = 3, 4, 5, \ldots$

$j+2 = n$
$j+1 = n-1$
$j-1 = n-3$

$a_2 = 0 \implies a_5 = 0, \ a_8 = 0, \ldots$

$a_3 = \dfrac{a_0}{3(2)}$, $a_6 = \dfrac{a_3}{6(5)} = \dfrac{a_0}{(3\cdot 2)(6\cdot 5)}$, $a_9 = \dfrac{a_6}{9\cdot 8} = \dfrac{a_0}{(9\cdot 8)(6\cdot 5)(3\cdot 2)}$

$a_4 = \dfrac{a_1}{4\cdot 3}$, $a_7 = \dfrac{a_4}{7\cdot 6} = \dfrac{a_1}{(7\cdot 6)(4\cdot 3)}$, $a_{10} = \dfrac{a_7}{10(9)} = \dfrac{a_1}{(10)(9)(7\cdot 6)(4\cdot 3)}$

Thus, $a_{3n} = \dfrac{a_0}{(3n)(3n-1)\cdots(6\cdot 5)(3\cdot 2)}$

$$\boxed{y = a_0 \left(\sum_{n=0}^{\infty} \frac{1}{(3n)(3n-1)\cdots(6\cdot 5)(3\cdot 2)} x^{3n} \right) + a_1 \left(\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n)\cdots(7\cdot 6)(4\cdot 3)} x^{3n+1} \right)}$$

grader: beware, there may be way to simplify the formulas ...