P9 # 25 on pg. 450

$$a_0 = 2 \qquad a_1 = 0$$

$$w'' + 3xw' - w = 0, \quad \overbrace{w(0) = 2}, \quad \overbrace{w'(0) = 0}$$

Set $w = 2 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$$w' = 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$$

$$\begin{aligned} w'' + 3xw' - w = {} & 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots \\ & + 3x(2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots) \\ & - (2 + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots) = 0 \end{aligned}$$

Thus,

x: $6a_3 = 0 \quad \therefore \underline{a_3 = 0}$.

x^2: $12a_4 + 6a_2 - a_2 = 0 \Rightarrow \underline{12a_4 + 5a_2 = 0}$.

x^3: $20a_5 + 9a_3 - a_3 = 0 \Rightarrow \underline{20a_5 + 8a_3 = 0}. \Rightarrow \underline{a_5 = 0}$.

x^4: $30a_6 + 12a_4 - a_4 \Rightarrow \underline{30a_6 + 11a_4 = 0}$.

$x^0 = 1$: $2a_2 - 2 = 0 \quad \therefore \underline{a_2 = 1}$

$\hookrightarrow 12a_4 = -5 \quad \therefore \underline{a_4 = \frac{-5}{12}}$.

$$a_6 = \frac{-11a_4}{30} = \frac{-11}{30}\left(\frac{-5}{12}\right) = \frac{11}{72} = a_6$$

Thus, $\boxed{w = 2 + x^2 - \frac{5}{12}x^4 + \frac{11}{72}x^6 + \cdots}$

PROBLEM 11 center power series of $f(x) = x\sin(x)$ at $x_0 = 3$.

$$
\begin{aligned}
f(x) &= x\sin(x) \\
&= \left[(x-3)+3\right]\left[\sin\left[(x-3)+3\right]\right] \\
&= \left((x-3)+3\right)\left(\sin(x-3)\cos(3) + \cos(x-3)\sin(3)\right) \\
&= (x-3)\cos(3)\sin(x-3) + \sin(3)(x-3)\cos(x-3) + 3\cos(3)\sin(x-3) + 3\sin(3)\cos(x-3) \\
&= \left[(x-3)\cos(3) + 3\cos(3)\right]\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x-3)^{2n+1}\right] + \left[(x-3)\sin(3) + 3\sin(3)\right]\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(x-3)^{2k}\right] \\
&= \boxed{\sum_{n=0}^{\infty}\left(\frac{(-1)^n\cos(3)}{(2n+1)!} + \frac{3\sin(3)(-1)^n}{(2n)!}\right)(x-3)^{2n} + \sum_{k=0}^{\infty}\left(\frac{(-1)^k 3\cos(3)}{(2k+1)!} + \frac{(-1)^k\sin(3)}{(2k)!}\right)(x-3)^{2k+1}}
\end{aligned}
$$

(I simply grouped even and odd terms together making the needed changes in notation for the grouping once I decided on $(x-3)^{2n}$ & $(x-3)^{2k+1}$ as my powers format)

PROBLEM 12 $f(x) = \dfrac{x^2}{2+x}$ find power series centered at $x_0 = -1$

$$
\begin{aligned}
f(x) &= \frac{x^2}{1+(x+1)} = \frac{(x+1)^2 - 2(x+1) + 1}{1+(x+1)} \qquad (x+1-1)^2 = (x+1)^2 - 2(x+1) + 1 \\
&= (x+1)^2\sum_{n=0}^{\infty}(-1)^n(x+1)^n - 2(x+1)\sum_{n=0}^{\infty}(-1)^n(x+1)^n + \sum_{n=0}^{\infty}(-1)^n(x+1)^n \\
&= \underline{\sum_{n=0}^{\infty}(-1)^n(x+1)^{n+2} - 2\sum_{n=0}^{\infty}(-1)^n(x+1)^{n+1} + \sum_{n=0}^{\infty}(-1)^n(x+1)^n}
\end{aligned}
$$

(this we can clean-up, see ↷)

$\boxed{p\ 12}$ continued

$$f(x) = -2 + 1 - (x+1) + \sum_{n=0}^{\infty} (-1)^n (x+1)^{n+2} \quad j = n+2$$

$$+ \sum_{n=1}^{\infty} 2(-1)^{n+1}(x+1)^{n+1} + \sum_{n=2}^{\infty} (-1)^n (x+1)^n \quad j=n+1 \quad j = n$$

$$= -1 - (x+1) + \sum_{j=2}^{\infty} \left((-1)^{j-2} + 2(-1)^j + (-1)^j \right) (x+1)^j$$

$$= \boxed{-1 - (x+1) + \sum_{j=2}^{\infty} 4(-1)^j (x+1)^j}$$

PROBLEM 13

$$e^x e^y = \left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right)\left(\sum_{j=0}^{\infty} \frac{y^j}{j!}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{1}{k!} x^k\right)\left(\frac{1}{(n-k)!} y^{n-k}\right)$$

$$\frac{n!}{n!} = 1.$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad : \quad \underbrace{\binom{n}{k} = \frac{n!}{k!(n-k)!}}_{\text{binomial coeff.}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n \quad \text{binomial Thm}$$

$$= \underline{\exp(x+y)}.$$

Problem 14 Suppose $\sum_{k=0}^{\infty}(a_{2k}x^{2k}+b_{2k+1}x^{2k+1}) = e^x+\cos(x+2)$. Find explicit formulas for a_{2k} and b_{2k+1} via Σ-notation algebra.

$$e^x+\cos(x+2) = \cosh x + \sinh x + \cos(2)\cos(x) - \sin(2)\sin(x)$$

$$= \underbrace{\cosh(x) + \cos(2)\cos(x)}_{\text{even}} + \underbrace{\sinh(x) - \sin(2)\sin(x)}_{\text{odd}}$$

$$= \sum_{j=0}^{\infty}\left(\frac{1}{(2j)!} + \frac{(-1)^j\cos(2)}{(2j)!}\right)x^{2j} + \sum_{j=0}^{\infty}\left(\frac{1-\sin(2)(-1)^j}{(2j+1)!}\right)x^{2j+1}$$

$$= \sum_{j=0}^{\infty}\left(\frac{1+(-1)^j\cos(2)}{(2j)!}\right)x^{2j} + \sum_{j=0}^{\infty}\left(\frac{1-(-1)^j\sin(2)}{(2j+1)!}\right)x^{2j+1}$$

Compare against

$$\sum_{j=0}^{\infty}\left(a_{2j}x^{2j} + b_{2j+1}x^{2j+1}\right)$$

to see that

$$\boxed{a_{2j} = \frac{1+(-1)^j\cos(2)}{(2j)!}}$$

and,

$$\boxed{b_{2j+1} = \frac{1-(-1)^j\sin(2)}{(2j+1)!}}$$

Naturally, we could switch $j \longmapsto k$ and express the same.

Problem 15 Find a power series solution to the integrals below:

(a.) $\int \frac{x^3+x^6}{1-x^3}\,dx$

(b.) $\int x^8 e^{x^3+2}\,dx$

(a.) Note that $\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n}$ for $|x^3| < 1$.

Hence,

$$\int \frac{x^3+x^6}{1-x^3}\,dx = \int \left(x^3 \sum_{n=0}^{\infty} x^{3n} + x^6 \sum_{n=0}^{\infty} x^{3n} \right) dx$$

$$= \int \left(\sum_{n=0}^{\infty} x^{3n+3} + \sum_{n=0}^{\infty} x^{3n+6} \right) dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{x^{3n+4}}{3n+4} + \sum_{n=0}^{\infty} \frac{x^{3n+7}}{3n+7}}$$ Sloppy answer

$3j = 3n+3$

$3j+4 = 3n+7.$

$$= C + \frac{x^4}{4} + \sum_{n=1}^{\infty} \frac{x^{3n+4}}{3n+4} + \sum_{j=1}^{\infty} \frac{x^{3j+4}}{3j+4}$$

$$= \boxed{C + \frac{x^4}{4} + \sum_{n=1}^{\infty} \frac{2}{3n+4} x^{3n+4}}$$ ← better answer (earned bonus)

(b.) $\int x^8 e^{x^3+2}\,dx = e^2 \int x^8 \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}\,dx$

$$= e^2 \int \sum_{n=0}^{\infty} \frac{x^{3n+8}}{n!}\,dx$$

$$= \boxed{e^2 \sum_{n=0}^{\infty} \frac{x^{3n+9}}{(3n+9)n!} + C}$$

Problem 16 Calculate the 42^{nd}-derivative of $x^2\cos(x)$ at $x = 1$. (use power series techniques)

Let $f(x) = x^2 \cos x$ then by Taylor's Th$^{\underline{m}}$ $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$

So if we can expand $f(x)$ in $(x-1)$-powers we can read off $f^{(42)}(1)$ with relative ease.

$$f(x) = (x-1+1)^2 \cos(x-1+1)$$
$$= \left[(x-1)^2 + 2(x-1)+1\right]\Big[\underbrace{\cos(1)\cos(x-1)}_{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(x-1)^{2k}} - \underbrace{\sin(1)\sin(x-1)}_{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(x-1)^{2k}}\Big]$$

The coefficient of $(x-1)^{42}$ is found,

$$1\cdot\cos(1)\frac{(-1)^{20}}{(40)!} + 1\cdot\cos(1)\frac{(-1)^{21}}{(42)!} + 2\cdot(-\sin(1))\frac{(-1)^{20}}{(41)!} = \frac{f^{(42)}(1)}{(42)!}$$

$$\Rightarrow f^{(42)}(1) = \cos(1)\left[\frac{(42)!}{(40)!} - \frac{(42)!}{(42)!}\right] - 2\sin(1)\left[\frac{(42)!}{(41)!}\right]$$

$$\Rightarrow f^{(42)}(1) = \cos(1)\left[(42)(41)-1\right] - 2\sin(1)\left[42\right]$$

$$\Rightarrow \boxed{f^{(42)}(1) = 1721\cos(1) - 84\sin(1) \cong 859.18}$$

Problem 17 Find the complete power series solution of $y'' + x^2y' + 2xy = 0$ about the ordinary point $x = 0$. Your answer should include nice formulas for arbitrary coefficients in each of the fundamental solutions. You need to both set-up and solve the reccurrence relations as best you can.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad 2xy = \sum_{n=0}^{\infty} 2a_n x^{n+1} = 2a_0 x + \cdots$$

$$x^2 y' = x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n+1} = a_1 x^2 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

Hence,

$$y'' + x^2 y' + 2xy = 0$$

$$2a_2 + 6a_3 x + \underbrace{\sum_{n=4}^{\infty} n(n-1) a_n x^{n-2}}_{\text{Let } j+1 = n-2,\ n = j+3} + \sum_{n=1}^{\infty} n a_n x^{n+1} + 2a_0 x + \sum_{n=1}^{\infty} 2a_n x^{n+1} = 0$$

$$2a_2 + (6a_3 + 2a_0)x + \sum_{j=1}^{\infty} \left((j+3)(j+2) a_{j+3} + j a_j + 2a_j \right) x^{j+1} = 0$$

Hence,

$$2a_2 = 0 \quad \Rightarrow \quad \underline{a_2 = 0} \quad \Rightarrow \quad 3k+2 \in 3\mathbb{Z}+2 \text{ has } a_{3k+2} = 0$$

$$6a_3 + 2a_0 = 0 \quad \Rightarrow \quad \underline{a_3 = -\tfrac{1}{3} a_0}$$

Same pattern.

For $j = 1, 2, 3, \ldots$

$$a_{j+3} = \frac{-\cancel{(2+j)}\, a_j}{(j+3)\cancel{(j+2)}} = \left(\frac{-1}{j+3} \right) a_j$$

Observe,

$$a_6 = \frac{-1}{6} a_3 = \left(\frac{-1}{6}\right)\left(\frac{-1}{3}\right) a_0$$

$$a_9 = \frac{-1}{9} a_6 = \left(\frac{-1}{9}\right)\left(\frac{-1}{6}\right)\left(\frac{-1}{3}\right) a_0$$

$$a_{12} = \frac{-1}{12} a_9 = \left(\frac{-1}{12}\right)\left(\frac{-1}{9}\right)\left(\frac{-1}{6}\right)\left(\frac{-1}{3}\right) a_0 = \frac{(-1)^4}{4!} \frac{a_0}{3^4}$$

$$a_{3k} = (-1)^k \frac{a_0}{k!\, 3^k}$$

Likewise,

$$a_4 = \frac{-1}{4} a_1$$

$$a_7 = \left(\frac{-1}{7}\right)\left(\frac{-1}{4}\right) a_1$$

$$a_{11} = \left(\frac{-1}{11}\right)\left(\frac{-1}{7}\right)\left(\frac{-1}{4}\right) a_1$$

$$a_{3k+1} = \frac{(-1)^k a_1}{(3k+1)\cdots 11 (7)(4)}$$

$$\therefore \quad \boxed{y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, 3^k} x^{3k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)\cdots(7)(4)} x^{3k+1}}$$

Problem 18 (Ritger & Rose 7-2 problem 7 part c) Find the first four nonzero terms in the power series solution about zero for the initial value problem $(x+2)y'' + 3y = 0$ with $y(0) = 0$ and $y'(0) = 1$.

Let $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

Note, $y(0) = c_0 = 0$, $y'(0) = c_1 = 1$ hence,

$$y = x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$y' = 1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$y'' = 2c_2 + 6c_3 x + 12 c_4 x^2 + 20 c_5 x^3 + 30 c_6 x^4 + \cdots$$

Consider,

$$(x+2)y'' + 3y = 0$$

Provides,

$$(x+2)(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \cdots) +$$
$$+ 3(x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots) = 0$$

Therefore, equating coefficients of $1, x, x^2, \ldots$ yields:

const) $4c_2 = 0 \Rightarrow c_2 = 0$.

x) $12c_3 + 2c_2 + 3 = 0 \Rightarrow c_3 = \frac{-1}{4}$.

x^2) $6c_3 + 24c_4 + 3c_2 = 0 \Rightarrow c_4 = \frac{-6c_3}{24} = \frac{+6}{4(24)} = \frac{1}{16}$.

x^3) $12c_4 + 40c_5 + 3c_3 = 0$

$$c_5 = \frac{1}{40}\left[-3\left(\frac{-1}{4}\right) - 12\left(\frac{1}{16}\right)\right] = \frac{1}{40}\left[\frac{3}{4} - \frac{3}{4}\right] = 0.$$

x^4) $20c_5 + 60c_6 + 3c_4 = 0$

$$c_6 = \frac{1}{60}\left[\frac{-3}{16} - 20(0)\right] = \frac{-1}{20(16)} = \frac{-1}{320}$$

$$\therefore \boxed{y = x - \frac{1}{4}x^3 + \frac{1}{16}x^4 - \frac{1}{320}x^6 + \cdots}$$

Problem 19 (Ritger & Rose 7-2 problem 7 part d) Find the first four nonzero terms in the power series solution about zero for the initial value problem $y'' + \sin(x)y' + (x-1)y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

$$y'' + \sin(x)\, y' + (x-1)y = 0 , \quad \underline{y(0) = 1, \; y'(0) = 0}$$

$$y = 1 + C_2x^2 + C_3x^3 + C_4x^4 + \cdots$$
$$y' = 2C_2x + 3C_3x^2 + 4C_4x^3 + \cdots$$
$$y'' = 2C_2 + 6C_3x + 12C_4x^2 + \cdots$$

Up to x^2

$\sin x = x + \cdots$

$$2C_2 + 6C_3x + 12C_4x^2 + \cdots + x(2C_2x + \cdots) + (x-1)(1 + C_2x^2 + \cdots) = 0$$

Const/ $2C_2 - 1 = 0 \quad \therefore \quad \underline{C_2 = 1/2}$.

x/ $6C_3 + 1 = 0 \quad \therefore \quad \underline{C_3 = -1/6}$.

x^2/ $12C_4 + 2C_2 - C_2 = 0 \quad \therefore \quad \underline{C_4 = \frac{-1}{12} C_2 = \frac{-1}{24}}$.

Thus,

$$\boxed{y = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots}$$

Problem 20 Construct a differential equation with $y_1(x) = \frac{\sin(x)}{x}$ for $x \neq 0$ and $y_1(0) = 1$, $y_2(x) = x$ as its fundamental solution set. To accomplish this task do two tasks:

(a.) Argue from appropriate facts from the theory of determinants that $L[y] = \det \begin{bmatrix} y & y' & y'' \\ y_1 & y_1' & y_1'' \\ y_2 & y_2' & y_2'' \end{bmatrix}$ is a linear ODE with solutions y_1 and y_2.

(b.) calculate $L[y]$ explicitly as a linear ODE of the form $py'' + qy' + ry = 0$ where p, q, r are perhaps given as Taylor expansions about zero.

(a.) $L[y] = \underbrace{y(y_1'y_2'' - y_2'y_1'') - y'(y_1y_2'' - y_2y_1'') + y''(y_1y_2' - y_2y_1')}_{\text{clearly linear ODE}}$

with coefficients $y_1y_2' - y_2y_1'$, $-y_1y_2'' + y_2y_1''$, $y_1'y_2'' - y_2'y_1'$

To see $L[y_1] = 0$ and $L[y_2] = 0$ simply note that setting $y = y_1$ or $y = y_2$ gives $L[y]$ a repeated row hence by prop. of det. $L[y_1] = L[y_2] = 0$.

(b.) $y_1 = \frac{1}{x}\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{7!}x^6 + \cdots$

$y_1' = \sum \frac{(-1)^k 2k}{(2k+1)!} x^{2k-1} = \frac{-1}{3}x + \frac{1}{30}x^3 - \frac{1}{7 \cdot 120}x^5 + \cdots$

$y_1'' = \frac{-1}{3} + \frac{1}{10}x^2 - \frac{1}{42}x^4 + \cdots$

Note, $y_2 = x$, $y_2' = 1$, $y_2'' = 0$ thus,

$r = y_1'y_2'' - y_2'y_1'' = \frac{1}{3} - \frac{1}{10}x^2 + \frac{1}{42}x^4 + \cdots$

$q = -y_1y_2'' + y_2y_1'' = \frac{-x}{3} + \frac{1}{10}x^3 - \frac{1}{42}x^5 + \cdots$

$p = y_1y_2' - y_2y_1' = \left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 \cdots\right) - x\left(\frac{-1}{3}x + \frac{1}{30}x^3 + \cdots\right)$

Thus,

$$\boxed{\left(1 + \frac{1}{6}x^2 - \frac{1}{40}x^4 + \cdots\right)y'' + \left(\frac{-x}{3} + \frac{1}{10}x^3 + \cdots\right)y' + \left(\frac{1}{3} - \frac{1}{10}x^2 + \frac{1}{42}x^4 \cdots\right)y = 0}$$