Make sure you name is on each page and the assignment is stapled. Thanks and enjoy. These problems are worth 3pts a piece (this makes 45 total points here or which 5 are bonus!)

Problem 1 Introduce variables to reduce

$$y''' + 4y'' + 2y' + 6y = \tan(t)$$

to a system of three first order ODEs in matrix normal form $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$.

Problem 2 Introduce variables to reduce

$$y'' + 4ty' + 5y' = 0, \qquad w'' + 9e^{-t}w = 0$$

to a system of four first order ODEs in matrix normal form $\frac{d\vec{x}}{dt} = A\vec{x}$.

Problem 3 Linear independence (LI) of vector-valued functions $\{\vec{f}_j : I \subseteq \mathbb{R} \to \mathbb{R}^n \mid j = 1, \dots, k\}$ is defined in the same way as was previously discussed for real-valued functions. In particular, $\{\vec{f}_1, \dots, \vec{f}_k\}$ is LI on $I \subseteq \mathbb{R}$ if $c_1\vec{f}_1(t) + \cdots + c_k\vec{f}_k(t) = 0$ for all $t \in I$ implies $c_1 = 0, \dots, c_k = 0$. We can check LI of n such n-vector-valued functions without any further differentiation; in particular, if $\det[\vec{f}_1(t)|\cdots|\vec{f}_n(t)] \neq 0$ for all $t \in I \subseteq \mathbb{R}$ then $\{\vec{f}_1(t), \dots, \vec{f}_n(t)\}$ is LI on I. Show the following sets of vector-valued functions are LI on $\mathbb{R}$. (notice, my notation is that $(a, b) = [a, b]^T$, in other words, each of the expressions below has lists of column vectors.

(a.) $\{\, (e^t, e^t),\ (e^t, -e^t)\, \}$

(b.) $\{\, (\cos(t), -\sin(t)),\ (\sin(t), \cos(t))\, \}$,

(c.) $\{\, e^t\vec{u}_1,\ ,\ e^t(\vec{u}_2 + t\vec{u}_1)\ , e^t(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_3)\, \}$ given $\vec{u}_1 = (1, 0, 0), \vec{u}_2 = (0, 1, 1), \vec{u}_3 = (1, 1, 1)$.

Problem 4 Solve $x' = 7x + 3y$ and $y' = 3x + 7y$ by the eigenvector method.

Problem 5 Use the solution of the previous problem to solve $x' = 7x + 3y + 1$ and $y' = 3x + 7y + 2$ subject the initial condition $x(0) = 1$ and $y(0) = 2$.

Problem 6 Solve $x' = -3x - 5y$ and $y' = 3x + y$ with $x(0) = 4$ and $y(0) = 0$ by the eigenvector method.

Problem 7 Use your eigensolutions from the previous problem to calculate the matrix exponential of

$$A = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix}$$

Problem 8 Solve $x' = 7x + 3y + 4z,\ y' = 6x + 2y,\ z' = 5z$ by the eigenvector method.

Problem 9 Use technology to find e-values and e-vectors for each of the matrices below. If possible, use the solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$ derived from e-vectors to write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$. If not possible, explain why.

(a.) $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

(b.) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

(c.) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$

(d.) $A = \begin{bmatrix} -1 & -3 & -9 \\ 0 & 5 & 18 \\ 0 & -2 & -7 \end{bmatrix}$.

(e.) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Problem 10 Solve, via the complex eigenvector technique,

$$\frac{dx}{dt} = 4x + 2y$$
$$\frac{dy}{dt} = -x + 2y.$$

Problem 11 Suppose $(A - \lambda I)\vec{u}_1 = 0$ and $(A - \lambda I)\vec{u}_2 = \vec{u}_1$ where $\lambda = 3 + i\sqrt{2}$ and $\vec{u}_1 = [3+i, 4+2i, 5+3i, 6+4i]^T$ and $\vec{u_2} = [i, 1, 2, 3-i]^T$.

(a.) find a pair of complex solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$

(b.) extract four real solutions to write the general real solution (c_1, c_2, c_3, c_4 should be real in this answer)

Problem 12 Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Calculate $e^{\theta J}$ where $\theta \in \mathbb{R}$. Express your answer in terms of sine and cosine and relevant matrices.

Problem 13 Solve $x' = 2x + y$ and $y' = 2y$ by the method generalized eigenvectors.

Problem 14 Suppose $\vec{v}$ is an eigevector with eigenvalue λ for the real matrix A. Show A^2 also has e-vector $\vec{v}$. What is the e-value for $\vec{v}$ with respect to A^2.

Problem 15 Let D be a diagonal matrix with $d_1, d_2, \dots, d_n$ on the diagonal . In other words, D is a matrix with components $D_{ij} = \delta_{ij} d_i$. Show that e^D is a diagonal matrix with $(e^D)_{ij} = \delta_{ij} e^{d_i}$. We needed this fact to establish the magic formula.