Make sure you name is on each page and the assignment is stapled. Thanks and enjoy. These problems are worth 3pts a piece (this makes 45 total points here or which 5 are bonus!)

Problem 1 Introduce variables to reduce

$$y''' + 4y'' + 2y' + 6y = \tan(t)$$

to a system of three first order ODEs in matrix normal form $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$.

Problem 2 Introduce variables to reduce

$$y'' + 4ty' + 5y' = 0, \qquad w'' + 9e^{-t}w = 0$$

to a system of four first order ODEs in matrix normal form $\frac{d\vec{x}}{dt} = A\vec{x}$.

Problem 3 Linear independence (LI) of vector-valued functions $\{\vec{f}_j : I \subseteq \mathbb{R} \to \mathbb{R}^n \mid j = 1, \dots, k\}$ is defined in the same way as was previously discussed for real-valued functions. In particular, $\{\vec{f}_1, \dots, \vec{f}_k\}$ is LI on $I \subseteq \mathbb{R}$ if $c_1\vec{f}_1(t) + \cdots + c_k\vec{f}_k(t) = 0$ for all $t \in I$ implies $c_1 = 0, \dots, c_k = 0$. We can check LI of n such n-vector-valued functions without any further differentiation; in particular, if $\det[\vec{f}_1(t)|\cdots|\vec{f}_n(t)] \neq 0$ for all $t \in I \subseteq \mathbb{R}$ then $\{\vec{f}_1(t), \dots, \vec{f}_n(t)\}$ is LI on I. Show the following sets of vector-valued functions are LI on $\mathbb{R}$. (notice, my notation is that $(a, b) = [a, b]^T$, in other words, each of the expressions below has lists of column vectors.

(a.) $\{\,(e^t, e^t),\ (e^t, -e^t)\,\}$

(b.) $\{\,(\cos(t), -\sin(t)),\ (\sin(t), \cos(t))\,\}$,

(c.) $\{\,e^t\vec{u}_1,\ ,\ e^t(\vec{u}_2 + t\vec{u}_1)\ , e^t(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_3)\,\}$ given $\vec{u}_1 = (1, 0, 0), \vec{u}_2 = (0, 1, 1), \vec{u}_3 = (1, 1, 1)$.

Problem 4 Solve $x' = 7x + 3y$ and $y' = 3x + 7y$ by the eigenvector method.

Problem 5 Use the solution of the previous problem to solve $x' = 7x + 3y + 1$ and $y' = 3x + 7y + 2$ subject the initial condition $x(0) = 1$ and $y(0) = 2$.

Problem 6 Solve $x' = -3x - 5y$ and $y' = 3x + y$ with $x(0) = 4$ and $y(0) = 0$ by the eigenvector method.

Problem 7 Use your eigensolutions from the previous problem to calculate the matrix exponential of

$$A = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix}$$

Problem 8 Solve $x' = 7x + 3y + 4z,\ y' = 6x + 2y,\ z' = 5z$ by the eigenvector method.

Problem 9 Use technology to find e-values and e-vectors for each of the matrices below. If possible, use the solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$ derived from e-vectors to write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$. If not possible, explain why.

(a.) $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

(b.) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

(c.) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$

(d.) $A = \begin{bmatrix} -1 & -3 & -9 \\ 0 & 5 & 18 \\ 0 & -2 & -7 \end{bmatrix}$.

(e.) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Problem 10 Solve, via the complex eigenvector technique,

$$\frac{dx}{dt} = 4x + 2y$$
$$\frac{dy}{dt} = -x + 2y.$$

Problem 11 Suppose $(A - \lambda I)\vec{u}_1 = 0$ and $(A - \lambda I)\vec{u}_2 = \vec{u}_1$ where $\lambda = 3 + i\sqrt{2}$ and $\vec{u}_1 = [3+i, 4+2i, 5+3i, 6+4i]^T$ and $\vec{u_2} = [i, 1, 2, 3-i]^T$.

(a.) find a pair of complex solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$

(b.) extract four real solutions to write the general real solution (c_1, c_2, c_3, c_4 should be real in this answer)

Problem 12 Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Calculate $e^{\theta J}$ where $\theta \in \mathbb{R}$. Express your answer in terms of sine and cosine and relevant matrices.

Problem 13 Solve $x' = 2x + y$ and $y' = 2y$ by the method generalized eigenvectors.

Problem 14 Suppose $\vec{v}$ is an eigevector with eigenvalue λ for the real matrix A. Show A^2 also has e-vector $\vec{v}$. What is the e-value for $\vec{v}$ with respect to A^2.

Problem 15 Let D be a diagonal matrix with $d_1, d_2, \ldots, d_n$ on the diagonal . In other words, D is a matrix with components $D_{ij} = \delta_{ij} d_i$. Show that e^D is a diagonal matrix with $(e^D)_{ij} = \delta_{ij} e^{d_i}$. We needed this fact to establish the magic formula.

MISSION 7 SOLUTION

PROBLEM 1 Introduce variables to reduce

$$y''' + 4y'' + 2y' + 6y = \tan(t)$$

to a system of 1^{st} order ODEs in matrix normal form.

$$X_1 = y \longrightarrow X_1' = X_2$$
$$X_2 = y' \longrightarrow X_2' = X_3$$
$$X_3 = y''$$
$$X_3' = y''' = \tan(t) - 4y'' - 2y' - 6y = \tan(t) - 4X_3 - 2X_2 - 6X_1$$

$$\boxed{\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -2 & -4 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tan(t) \end{bmatrix}}$$

PROBLEM 2 Same as last problem, reduce order, find equivalent system in matrix normal form

$$y'' + 4ty' + 5y' = 0$$
$$w'' + 9e^{-t} w = 0$$

$$X_1 = y \longrightarrow X_1' = X_2$$
$$X_2 = y' \longrightarrow X_2' = y'' = (-4t+5)y' = (-4t+5)X_2$$
$$X_3 = w \longrightarrow X_3' = X_4$$
$$X_4 = w' \longrightarrow X_4' = w'' = -9e^{-t} w = -9e^{-t} X_3$$

$$\boxed{\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 5-4t & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -9e^{-t} & 0 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}}$$

Problem 3 Linear independence (LI) of vector-valued functions $\{\vec{f}_j : I \subseteq \mathbb{R} \to \mathbb{R}^n \mid j = 1, \dots, k\}$ is defined in the same way as was previously discussed for real-valued functions. In particular, $\{\vec{f}_1, \dots, \vec{f}_k\}$ is LI on $I \subseteq \mathbb{R}$ if $c_1\vec{f}_1(t) + \cdots + c_k\vec{f}_k(t) = 0$ for all $t \in I$ implies $c_1 = 0, \dots, c_k = 0$. We can check LI of n such n-vector-valued functions without any further differentiation; in particular, if $\det[\vec{f}_1(t)|\cdots|\vec{f}_n(t)] \neq 0$ for all $t \in I \subseteq \mathbb{R}$ then $\{\vec{f}_1(t), \dots, \vec{f}_n(t)\}$ is LI on I. Show the following sets of vector-valued functions are LI on $\mathbb{R}$. (notice, my notation is that $(a, b) = [a, b]^T$, in other words, each of the expressions below has lists of column vectors.

(a.) $\{ (e^t, e^t),\ (e^t, -e^t) \}$

(b.) $\{ (\cos(t), -\sin(t)),\ (\sin(t), \cos(t)) \}$,

(c.) $\{ e^t\vec{u}_1,\ ,\ e^t(\vec{u}_2 + t\vec{u}_1)\ , e^t(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_3) \}$ given $\vec{u}_1 = (1, 0, 0), \vec{u}_2 = \overbrace{(0, 1, 1)}^{(0,1,0)}, \vec{u}_3 = \overbrace{(1, 1, 1)}^{(0,0,1)}$.

(a.) $\det\begin{bmatrix} e^t & e^t \\ e^t & -e^t \end{bmatrix} = -2e^t \neq 0 \quad \forall t \in \mathbb{R} \therefore \left\{ \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \begin{bmatrix} e^t \\ -e^t \end{bmatrix} \right\}$ is LI on $\mathbb{R}$.

(b.) $\det\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \cos^2 t + \sin^2 t = 1$ thus,

$\{ (\cos t, -\sin t), (\sin t, \cos t) \}$ is LI on $\mathbb{R}$.

SORRY, LOOKS LIKE THIS ONE WAS FOR FREE...

(c.) $\det\left[e^t\vec{u}_1 \middle| e^t(\vec{u}_2 + t\vec{u}_1) \middle| e^t\left(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_3\right) \right] =$

$= (e^t)^3 \det\left[\vec{u}_1 | \vec{u}_2 | \vec{u}_3 \right]$ Via multilinearity of det and fact that $\det[\underbrace{\vec{u}_1 | t\vec{u}_1}_{\text{scalar multiple.}} | \cdots] = 0$

$= e^{3t} \underbrace{\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{\text{oops!}}$

oops! I think I made an error setting this one up.

$(e^t)^3 \det\begin{bmatrix} 1 & t & 1 + t^2/2 \\ 0 & 1 & 1 + t + t^2/2 \\ 0 & 1 & 1 + t + t^2/2 \end{bmatrix} = (e^t)^3 \left(1 + t + t^2/2 - (1 + t + t^2/2)\right) = 0.$

(just checking to make sure...)

Problem 4 Solve $x' = 7x + 3y$ and $y' = 3x + 7y$ by the eigenvector method.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{bmatrix} = (\lambda-7)^2 - 9$$
$$= (\lambda - 7 - 3)(\lambda - 7 + 3)$$
$$= (\lambda - 10)(\lambda - 4)$$
$$\therefore \lambda_1 = 10, \ \lambda_2 = 4$$

$\lambda_1 = 10$) $(A - 10I)\vec{u}_1 = \underbrace{\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}}_{\substack{u - v = 0 \\ \Rightarrow u = v}} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u - v = 0$, choose $v = 1$ $\rightarrow$ $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\Rightarrow u = v$

this yields $\vec{x}_1(t) = e^{10t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = 4$) $(A - 4I)\vec{u}_2 = \underbrace{\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}}_{u + v = 0} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u + v = 0 \Rightarrow u = -v$

choose $v = 1$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

this yields $\vec{x}_2(t) = e^{4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Consequently, the gen. sol$^{\underline{n}}$ is:

$$\boxed{\vec{x}(t) = c_1 e^{10t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

Problem 5 Use the solution of the previous problem to solve $x' = 7x + 3y + 1$ and $y' = 3x + 7y + 2$ subject the initital condition $x(0) = 1$ and $y(0) = 2$.

As I mentioned in lecture, undet. coeff actually works here w/o much trouble,

$$\vec{x}_p = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{need} \quad A\vec{x}_p + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{d\vec{x}_p}{dt} = 0$$

Hence solve $\begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{49-9}\begin{bmatrix} 7 & -3 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{40}\begin{bmatrix} -1 \\ -11 \end{bmatrix} = \frac{-1}{40}\begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Checking: $A\begin{bmatrix} 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 7+33 \\ 3+77 \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \end{bmatrix} \Rightarrow A\vec{x}_p = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ ✓

Thus

$$\vec{x}(t) = c_1 e^{10t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/40 \\ -11/40 \end{bmatrix}$$

Now fit initial conditions

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/40 \\ -11/40 \end{bmatrix}$$

arranged as matrix eq^n

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 41/40 \\ 91/40 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{1+1}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 41/40 \\ 91/40 \end{bmatrix} = \frac{1}{2(40)}\begin{bmatrix} 41+91 \\ -41+91 \end{bmatrix} = \frac{1}{80}\begin{bmatrix} 132 \\ 50 \end{bmatrix}$$

$$\boxed{\vec{x}(t) = \frac{132}{80}e^{10t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{50}{80}e^{4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/40 \\ 11/40 \end{bmatrix}}$$

of course you can reduce fractions, combine terms etc...

$\boxed{\text{PROBLEM 6}}$ $x' = -3x - 5y \quad x(0) = 4$

$y' = 3x + y \quad y(0) = 0$

Solve via e-vector method

$$A = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix} \longrightarrow \det\begin{bmatrix} -3-\lambda & -5 \\ 3 & 1-\lambda \end{bmatrix} = (\lambda-1)(\lambda+3) + 15$$

$$= \lambda^2 + 2\lambda + 12$$

$$= (\lambda+1)^2 + 11$$

Thus $\lambda = -1 \pm i\sqrt{11}$

Study $\lambda = -1 + i\sqrt{11}$

$$(A - \lambda I)\vec{u} = \left[\begin{array}{c|c} -3+1-i\sqrt{11} & -5 \\ \hline 3 & 1+1-i\sqrt{11} \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3u + (2 - i\sqrt{11})v = 0$$

Set $\underline{v = 3}$. then $3u = (i\sqrt{11} - 2)3 \therefore \underline{u = i\sqrt{11} - 2}$.

$$\vec{u} = \begin{bmatrix} -2 + i\sqrt{11} \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}_{\vec{a}} + i\underbrace{\begin{bmatrix} \sqrt{11} \\ 0 \end{bmatrix}}_{\vec{b}}$$

Hence $\vec{x}(t) = c_1 \mathrm{Re}\left(e^{(-1+i\sqrt{11})t}(\vec{a} + i\vec{b})\right) + c_2 \mathrm{Im}\left(e^{(-1+i\sqrt{11})t}(\vec{a}+i\vec{b})\right)$

$$\vec{x}(t) = c_1 e^{-t}\left(\cos(t\sqrt{11})\begin{bmatrix} -2 \\ 3 \end{bmatrix} - \sin(\sqrt{11}t)\begin{bmatrix} \sqrt{11} \\ 0 \end{bmatrix}\right) +$$

$$+ c_2 e^{-t}\left(\sin(\sqrt{11}t)\begin{bmatrix} -2 \\ 3 \end{bmatrix} + \cos(\sqrt{11}t)\begin{bmatrix} \sqrt{11} \\ 0 \end{bmatrix}\right)$$

Then $\vec{x}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} \sqrt{11} \\ 0 \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2\sqrt{11} \\ 3c_1 \end{bmatrix}$

$\therefore c_1 = 0$ and $c_2\sqrt{11} = 4 \therefore c_2 = 4/\sqrt{11}$

$$\therefore \boxed{\vec{x}(t) = \frac{4}{\sqrt{11}} e^{-t}\begin{bmatrix} -2\cos(t\sqrt{11}) - \sqrt{11}\sin(t\sqrt{11}) \\ 3\cos(t\sqrt{11}) \end{bmatrix}}$$

PROBLEM 7 Find e^A for $A = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix}$

Recall $e^A \vec{u} = e^{\lambda} \vec{u}$ (or prove it for yourself it's not too hard...)

$\lambda_1 = -1 + i\sqrt{11}$ has $\vec{u}_1 = (-2 + i\sqrt{11}, 3)$

$\lambda_2 = -1 - i\sqrt{11}$ has $\vec{u}_2 = (-2 - i\sqrt{11}, 3)$

Thus,

$$e^A \vec{u}_1 = e^{-1+i\sqrt{11}} \vec{u}_1 = \frac{1}{e}\left(\cos\sqrt{11} + i\sin\sqrt{11}\right)\vec{u}_1$$

$$e^A \vec{u}_2 = e^{-1-i\sqrt{11}} \vec{u}_2 = \frac{1}{e}\left(\cos\sqrt{11} - i\sin\sqrt{11}\right)\vec{u}_2$$

From which we find,

$$e^A [\vec{u}_1 | \vec{u}_2] = \frac{1}{e}\left[e^{i\sqrt{11}} \vec{u}_1 \,\middle|\, e^{-i\sqrt{11}} \vec{u}_2\right]$$

$\begin{bmatrix} -2+i\sqrt{11} & -2-i\sqrt{11} \\ 3 & 3 \end{bmatrix}^{-1}$ 2x2 inverse formula.

$$\Rightarrow \quad e^A = \frac{1}{e}\left[e^{i\sqrt{11}} \vec{u}_1 \,\middle|\, e^{-i\sqrt{11}} \vec{u}_2\right]\left[\vec{u}_1 \,\middle|\, \vec{u}_2\right]^{-1}$$

$$= \frac{1}{e}\left[e^{i\sqrt{11}}\begin{bmatrix} -2+i\sqrt{11} \\ 3 \end{bmatrix} \middle| \, e^{-i\sqrt{11}}\begin{bmatrix} -2-i\sqrt{11} \\ 3 \end{bmatrix}\right] \frac{1}{3(i\sqrt{11}-2)+3(i\sqrt{11}+2)}\begin{bmatrix} 3 & 2+i\sqrt{11} \\ -3 & -2+i\sqrt{11} \end{bmatrix}$$

$$= \frac{1}{6ei\sqrt{11}}\left[e^{i\sqrt{11}}\begin{bmatrix} -2+i\sqrt{11} \\ 3 \end{bmatrix} \middle| \, e^{-i\sqrt{11}}\begin{bmatrix} -2-i\sqrt{11} \\ 3 \end{bmatrix}\right]\begin{bmatrix} 3 & 2+i\sqrt{11} \\ -3 & -2+i\sqrt{11} \end{bmatrix}$$

$$(*) = \frac{1}{6ei\sqrt{11}}\left[\begin{array}{c|c} e^{i\sqrt{11}}(-2+i\sqrt{11})(3) + e^{-i\sqrt{11}}(-2-i\sqrt{11})(-3) & \\ \hline 3e^{i\sqrt{11}}(2+i\sqrt{11}) + e^{-i\sqrt{11}}3(-2+i\sqrt{11}) & \end{array}\right]$$

$$\vdots$$

$$\cong \left[\begin{array}{c|c} -0.3236 & 0.0966 \\ \hline -0.0579 & -0.4009 \end{array}\right]$$

for some details.

$$\boxed{= \frac{1}{11e}\left[\begin{array}{c|c} 11\cos\sqrt{11} - 2\sin\sqrt{11} & -5\sqrt{11}\sin\sqrt{11} \\ \hline 3\sqrt{11}\sin(\sqrt{11}) & 11\cos\sqrt{11} + 2\sqrt{11}\sin\sqrt{11} \end{array}\right]}$$

continued,

$$e^{i\sqrt{11}}\left(-2+i\sqrt{11}\right)(3)+e^{-i\sqrt{11}}\left(2+i\sqrt{11}\right)(3) =$$

$$= \left(\cos\sqrt{11}+i\sin\sqrt{11}\right)\left(-6+i\,3\sqrt{11}\right)+\left(\cos\sqrt{11}-i\sin\sqrt{11}\right)\left(6+3i\sqrt{11}\right)$$

$$= \cancel{-6\cos\sqrt{11}} \cancel{-3\sqrt{11}\sin\sqrt{11}} \cancel{+6\cos\sqrt{11}} \cancel{+3\sqrt{11}\sin\sqrt{11}} +$$

$$+\,i\left(-6\sin\sqrt{11}+3\sqrt{11}\cos\sqrt{11}+\cancel{6(\cos\sqrt{11})\sqrt{11}}\ 3\sqrt{11}\cos\sqrt{11}-6\sin\sqrt{11}\right)$$

$$= i\left(-12\sin\sqrt{11}+6\sqrt{11}\cos\sqrt{11}\right)$$

then we have $\dfrac{1}{6ei\sqrt{11}}$ in front of matrix at $(*)$

hence $$\left(e^{A}\right)_{11} = \frac{1}{6ei\sqrt{11}}\, i\left(-12\sin\sqrt{11}+6\sqrt{11}\cos\sqrt{11}\right)$$

$$= -\frac{2\sin\sqrt{11}}{e\sqrt{11}}+\frac{\cos\sqrt{11}}{e}$$

$$= \frac{1}{11e}\left(11\cos\sqrt{11}-2\sin\sqrt{11}\right)$$

Similar tedious arithmetic should affirm my answer's $(1,2)$, $(2,1)$, $(2,2)$ entries for e^{A}.

Problem 8 Solve $x' = 7x + 3y + 4z,\ y' = 6x + 2y,\ z' = 5z$ by the eigenvector method.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \underbrace{\begin{bmatrix} 7 & 3 & 4 \\ 6 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} 7-\lambda & 3 & 4 \\ 6 & 2-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$= (5-\lambda)\left[(7-\lambda)(2-\lambda) - 18\right]$$

$$= (5-\lambda)\left[(\lambda-7)(\lambda-2) - 18\right]$$

$$= (5-\lambda)\left[\lambda^2 - 9\lambda + 14 - 18\right]$$

$$= (5-\lambda)\left[\lambda^2 - 9\lambda - 4\right] = (5-\lambda)\underbrace{(\lambda + 0.4244)(\lambda - 9.4244)}_{\text{approximate.}}$$

It follows, I used technology as the $\lambda_2 = -0.4244,\ \lambda_3 = 9.4244$ are ugly.

$$\boxed{\vec{x}(t) = c_1 e^{5t}\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} + c_2 e^{-0.4244t}\begin{bmatrix} -0.37 \\ 0.93 \\ 0 \end{bmatrix} + c_3 e^{9.4244t}\begin{bmatrix} 0.78 \\ 0.63 \\ 0 \end{bmatrix}}$$

Problem 9 Use technology to find e-values and e-vectors for each of the matrices below. If possible, use the solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$ derived from e-vectors to write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$. If not possible, explain why.

(a.) $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, $\longrightarrow \vec{x}(t) = c_1 e^{-t}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{8t}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

(b.) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $\longrightarrow \lambda_1 = \lambda_2 = 3$, $\lambda_3 = 5$, only two-e-vectors thus we cannot write gen. sol$^{\underline{n}}$ just with e-vectors. Need e^{At} etc... later.

(c.) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$, $\longrightarrow \vec{x}(t) = c_1 e^{t}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is all we get here.

$\lambda_1 = 1 = \lambda_2 = \lambda_3$ but only <u>one</u> e-vector.

(d.) $A = \begin{bmatrix} -1 & -3 & -9 \\ 0 & 5 & 18 \\ 0 & -2 & -7 \end{bmatrix}$, $\lambda_1 = \lambda_2 = \lambda_3 = -1$ but only <u>two</u> LI e-vectors exist for this A so, again we cannot find 3 LI e-vector sol$^{\underline{n}}$s.

(e.) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$$y = c_1 e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{t}\left(\cos t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) + c_3 e^{t}\left(\sin t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \cancel{\cos t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right),$$

and

cos t.

Problem 10 Solve, via the complex eigenvector technique,

$$\frac{dx}{dt} = 4x + 2y \qquad \longrightarrow \vec{x}' = A\vec{x} \quad \text{for} \quad A = \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}$$
$$\frac{dy}{dt} = -x + 2y.$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & 2 \\ -1 & 2-\lambda \end{pmatrix} = (\lambda-4)(\lambda-2) + 2$$
$$= \lambda^2 - 6\lambda + 8 + 2$$
$$= (\lambda-3)^2 + 1 = 0$$
$$\underline{\lambda = 3 \pm i}.$$

Choose (+) to find e-vector,

$$(A - (3+i)I)\vec{u} = \begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{array}{l} -u = (1+i)v \\ u = -(1+i)v \end{array}$$

Choose $v = 1 \Rightarrow u = -1 - i$

$$\vec{u} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{"}\vec{a}\text{"}} + i\underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{\text{"}\vec{b}\text{"}}$$

As usual, select real & imaginary components of the complex sol^{n} $e^{3t}(\cos t + i\sin t)(\vec{a} + i\vec{b})$

$$\boxed{\vec{x}(t) = c_1 e^{3t}\left(\cos t\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + c_2 e^{3t}\left(\sin t\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \cos t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)}$$

Of course, the selection of $\vec{u}$ can be replaced with $c\vec{u}$ for any $c \in \mathbb{C}$ with $c \neq 0$ hence the sol^{n}'s appearance is far from unique, but the ambiguity is covered by the freedom to select c_1 & c_2 arbitrarily.

Problem 11. Suppose $(A-\lambda I)\vec{u}_1 = 0$ and $(A-\lambda I)\vec{u}_2 = \vec{u}_1$ where $\lambda = 3+i\sqrt{2}$ and $\vec{u}_1 = [3+i, 4+2i, 5+3i, 6+4i]^T$ and $\vec{u_2} = [i, 1, 2, 3-i]^T$.

(a.) find a pair of complex solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$

$$\vec{z}_1 = e^{At}\vec{u}_1 = e^{\lambda t}(I + t(A-\lambda I) + \cdots)\vec{u}_1 = e^{\lambda t}\vec{u}_1$$

$$\vec{z}_2 = e^{At}\vec{u}_2 = e^{\lambda t}(\vec{u}_2 + t(A-\lambda I)\vec{u}_2 + \cdots) = e^{\lambda t}(\vec{u}_2 + t\,\vec{u}_1)$$

$$\boxed{\vec{z}_1 = e^{3t}e^{i\sqrt{2}t}\begin{bmatrix} 3+i \\ 4+2i \\ 5+3i \\ 6+4i \end{bmatrix}} \quad \& \quad \boxed{\vec{z}_2 = e^{3t}e^{i\sqrt{2}t}\left(\begin{bmatrix} i \\ 1 \\ 2 \\ 3-i \end{bmatrix} + t\begin{bmatrix} 3+i \\ 4+2i \\ 5+3i \\ 6+4i \end{bmatrix}\right)}$$

(b.) extract four real solutions to write the general real solution (c_1, c_2, c_3, c_4 should be real in this answer)

$$\vec{X}(t) = c_1 Re(\vec{z}_1) + c_2 Im(\vec{z}_1) + c_3 Re(\vec{z}_2) + c_4 Im(\vec{z}_2)$$

$$= c_1 e^{3t}\left[\cos t\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \sin t\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right] + c_2 e^{3t}\left[\sin t\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} + \cos t\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right] +$$

$$+ c_3 e^{t}\left[\cos t\begin{bmatrix} 3t \\ 1+4t \\ 2+5t \\ 3+6t \end{bmatrix} - \sin t\begin{bmatrix} 1+t \\ 2t \\ 3t \\ -1+4t \end{bmatrix}\right] + c_4 e^{t}\left[\sin t\begin{bmatrix} 3t \\ 1+4t \\ 2+5t \\ 3+6t \end{bmatrix} + \cos t\begin{bmatrix} 1+t \\ 2t \\ 3t \\ -1+4t \end{bmatrix}\right]$$

Problem 12 Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Calculate $e^{\theta J}$ where $\theta \in \mathbb{R}$. Express your answer in terms of sine and cosine and relevant matrices.

$$J^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$J^3 = JJ^2 = -JI = -J$$

$$J^4 = J^2J^2 = (-I)(-I) = I.$$

$$J^{2k} = (-1)^k I$$

$$J^{2j+1} = (-1)^j J$$

Thus,

$$e^{\theta J} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} J^n \quad \text{even} \quad \text{odd}$$

$$= \sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!} J^{2k} + \sum_{j=0}^{\infty} \frac{\theta^{2j+1}}{(2j+1)!} J^{2j+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} I + \sum_{j=0}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} J$$

$$= \underbrace{\left(\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}\right)}_{\cos\theta} I + \underbrace{\left(\sum_{j=0}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!}\right)}_{\sin\theta} J$$

$$\therefore \boxed{e^{\theta J} = (\cos\theta) I + (\sin\theta) J}$$

Problem 13 Solve $x' = 2x + y$ and $y' = 2y$ by the method generalized eigenvectors.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \det(A - \lambda I) = (\lambda - 2)^2 = 0$$

$$\therefore \lambda_1 = 2 \text{ twice.}$$

$$(A - 2I)\vec{u}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{} \begin{array}{c} v = 0 \\ u \text{ free} \\ \text{Choose } u = 1 \end{array} \to \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}.$$

$$(A - 2I)\vec{u}_2 = \vec{u}_1 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{} \begin{array}{c} v = 1 \\ u \text{ free} \\ \text{choose} \\ u = 0 \end{array} \to \underline{\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}.$$

Hence,

$$\boxed{\vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}$$

Of course, we derive these from the matrix exponential (which we proved is a sol$^{\underline{n}}$ matrix) and the identity

$$e^{At} = e^{2t}\left[I + t(A - 2I) + \frac{t^2}{2}(A - 2I)^2 + \cdots\right]$$

It's clear that

$$e^{At}\vec{u}_1 = e^{2t}\vec{u}_1$$

$$e^{At}\vec{u}_2 = \underline{e^{2t}(\vec{u}_2 + t\vec{u}_1)}. \quad *$$

<u>Remark</u>: sometimes on tests I'll number vectors a little different to make sure you're not just using * w/o thinking...

PROBLEM 14

Suppose $\lambda \in \mathbb{R}$ and $A\vec{v} = \lambda \vec{v}$ for some $\vec{v} \neq 0$. Show A^2 also has $\vec{v}$ as an e-vector. What is e-value of $\vec{v}$ for A^2?

$$A^2 \vec{v} = A A \vec{v} = A \lambda \vec{v} = \lambda A \vec{v} = \lambda \lambda \vec{v} = \lambda^2 \vec{v}$$

thus A^2 has e-vector $\vec{v}$ with e-value λ^2.

PROBLEM 15

Let $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$

observe $D^k = \begin{bmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{bmatrix}$ (this can be proven by induction on k)

Thus

$$e^D = \sum_{k=0}^{\infty} \frac{D^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} d_1^k & & & \\ & \sum_{k=0}^{\infty} \frac{1}{k!} d_2^k & & \\ & & \ddots & \\ & & & \sum_{k=0}^{\infty} \frac{1}{k!} d_n^k \end{bmatrix}$$

$$= \begin{bmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{bmatrix}.$$

For example, $e^{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} = \begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix}$