SPRINGS WITH DAMPING: (§4.1 & §4.9)

Consider a mass moving thru a viscous media, a good **model** to the friction force is $-b\frac{dx}{dt} = F_{friction}$, as before $-kx = F_{spring}$ thus in the absence of other forces Newton's 2nd Law reads:

$$\boxed{ma = -bv - kx}$$ (note $m, b, k > 0$ for physical reasons)

Which is using $\dot{x} = \frac{dx}{dt} = v$ & $\ddot{x} = \frac{d^2x}{dt^2} = a$ simply an 2nd order Linear ODEq^n with constant coefficients,

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0 \quad Eq^n (1)}$$

We then have the characteristic Eq^n: $m\lambda^2 + b\lambda + k = 0$ with sol^n's

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

Which depending on the values of m, b, k gives sol^n's to Eq^n (1)

PHYSICAL DESCRIPTION	FORM OF SOL^n	Condition on m, b, k:
OVER DAMPING	$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$	I.) $b^2 - 4mk > 0$
CRITICAL DAMPING	$x = c_1 e^{\lambda t} + c_2 x e^{\lambda t}$	II.) $b^2 - 4mk = 0$
UNDER DAMPING	$x = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$	III.) $b^2 - 4mk < 0$; $\alpha = \frac{-b}{2m}$ $\beta = \frac{\sqrt{4mk - b^2}}{2m}$

E94 Suppose we have a 5 kg mass secured to a spring with $k = 1($ if we place the spring & mass in a liquid with $c = 2$ kg/s, describe the resulting motion.

$$5\ddot{x} + 2\dot{x} + 10x = 0 \Rightarrow 5\lambda^2 + 2\lambda + 10 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 200}}{10} = \frac{-1}{5} \pm i\frac{\sqrt{196}}{10} = \frac{-1}{5} \pm$$

Thus $x(t) = e^{-0.2t}\left(c_1 \cos(1.4t) + c_2 \sin(1.4t)\right)$. Further suppose that the mass started at the equilibrium position with velocity $\dot{x}(0) = 1.4$ find the eq^n of motion. First find the velocity at time t,

$$\dot{x} = \frac{dx}{dt} = -0.2e^{-0.2t}\left(c_1\cos(1.4t) + c_2\sin(1.4t)\right) + e^{-0.2t}\left(-1.4c_1\sin(1.4t) + 1.4c_2\cos(1.4t)\right)$$

Now use the intial conditions;

$$x(0) = e^0(c_1\cos(0) + c_2\sin(0)) = \boxed{c_1 = 0}$$

$$\dot{x}(0) = -0.2e^0(\underbrace{c_1}_{0}\cos(0) + c_2\underbrace{\sin(0)}_{0}) + e^0(-1.4c_1\underbrace{\sin(0)}_{0} + 1.4c_2\cos(0)) = 1.4c_2 = 1.$$

$\therefore \boxed{c_2 = 1}$ $\quad \therefore \boxed{x(t) = e^{-0.2t}\sin(1.4t)}$ ← Eq^n of Motion

Concerning the form of the sol$^{\underline{n}}$ of $m\ddot{x} + b\dot{x} + kx = 0$

We wrote the general sol$^{\underline{n}}$ of the underdamped case where $b^2 - 4mk < 0$ and $\alpha = -b/2m$, $\beta = \sqrt{4mk - b^2}$ as

$$\boxed{x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)} \text{ — I}$$

This is correct, but sometimes it is convenient to write the sol$^{\underline{n}}$ in terms of an single sine function.

$$\boxed{x(t) = A e^{\alpha t} \sin(\beta t + \phi)} \text{ — II}$$

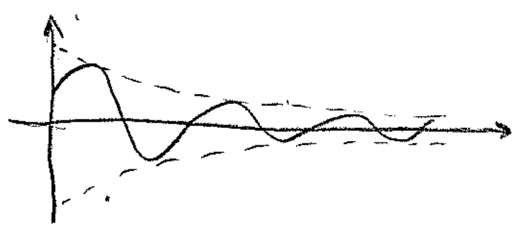
To connect I and II we need to recall the adding-angles formula for $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ thus,

$$\begin{aligned} x(t) &= Ae^{\alpha t}(\sin \beta t \cos \phi + \sin \phi \cos \beta t) \\ &= \underline{A\cos\phi}\ \underline{e^{\alpha t} \sin \beta t} + \underline{A\sin\phi}\ \underline{e^{\alpha t} \cos \beta t} = \underline{c_1} \underline{e^{\alpha t} \cos \beta t} + \underline{c_2} \underline{e^{\alpha t} \sin \beta t} \end{aligned}$$

Comparing these we find the arbitrary constants A, ϕ and c_1, c_2 are related via

$$\boxed{\begin{array}{lcl} c_1 = A\sin\phi & & \phi = \tan^{-1}(c_1/c_2) \\ c_2 = A\cos\phi & & A^2 = c_1^2 + c_2^2 \end{array}}$$

It's easy to graph $x(t) = Ae^{\alpha t}\sin(\beta t + \phi)$



$Ae^{\alpha t}$ serves as a decreasing "amplitude" of this function with "frequency" $\frac{\beta}{2\pi} = \frac{\sqrt{4mk - b^2}}{4m\pi}$

Technically it's a "quasifrequency" since this is not really a periodic function. Truly periodic functions have the property $f(t) = f(t+T)$ for all $t \in \text{dom}(f)$ and we call T the period of the <u>periodic function</u> f

- If $b = 0$ then we have pure harmonic oscillation. In this case A is a genuine amplitude and the frequency is $\frac{\beta}{2\pi} = \frac{\sqrt{4mk}}{4m\pi} = \frac{1}{2\pi}\sqrt{k/m}$.

Forced Oscillations: (§ 4.10)

Suppose that in addition to the damping force we have some motor pushing & pulling on the spring; $F(t)$ then we have Newton's 2^{nd} Law:

$$ma = -kx - b\dot{x} + F(t)$$

Which gives a non-homogeneous 2^{nd} order ODEq^{n}:

$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t) \quad : \quad Eq^{n}(2)}$$

Now for certain functions $F(t)$ we have the technology to solve this.

E95 Let $m = 1kg$ and $k = 25N/m$ with $b = 0$ and force $F(t) = F_0\cos(5t)$

$$\ddot{x} + 25x = F_0\cos(5t)$$

Find the resulting motion if $x(0) = 10$ and $\dot{x}(0) = 0$.

$$\lambda^2 + 25 = 0 \Rightarrow \lambda = \pm 5i \Rightarrow x_h(t) = c_1\cos(5t) + c_2\sin(5t)$$

Notice x_h overlaps the forcing term so guess: $x_p = t(A\cos(5t) + B\sin(5t))$

$$\dot{x}_p = A\cos(5t) + B\sin(5t) + t(-5A\sin(5t) + 5B\cos(5t))$$
$$= \cos(5t)[A + 5Bt] + \sin(5t)[B - 5At]$$

$$\ddot{x}_p = -5\sin(5t)[A + 5Bt] + 5B\cos(5t)$$
$$+ 5\cos(5t)[B - 5At] - 5A\sin(5t)$$
$$= \cos(5t)[10B - 25At] + \sin(5t)[-10A - 25Bt]$$

Substituting into $\ddot{x}_p + 25x_p = F_0\cos(5t)$ to determine A, B gives,

$$\cos(5t)[10B - 25At] + \sin(5t)[-10A + 25Bt] + 25t(A\cos(5t) + B\sin(5t)) = F_0\cos(5t)$$

$$\left.\begin{aligned} \cos(5t) &: \quad 10B = F_0 \\ \sin(5t) &: \quad -10A = 0 \end{aligned}\right\} \Rightarrow A = 0 \ \& \ B = F_0/10$$

$$\left.\begin{aligned} t\cos(5t) &: \quad -25A + 25A = 0 \\ t\sin(5t) &: \quad 25B - 25B = 0 \end{aligned}\right\} \text{ no info.}$$

The general sol^{n} is thus $x(t) = c_1\cos(5t) + c_2\sin(5t) + \frac{F_0}{10}\cos(5t)\cdot t$

Then $\dot{x}(t) = -5c_1\sin(5t) + 5c_2\cos(5t) - \frac{F_0}{2}\sin(5t) + \frac{F_0}{10}\cos(5t)$

$$x(0) = c_1 + F_0/10 = 10 \Rightarrow \boxed{c_1 = 10 - F_0/10}$$
$$\dot{x}(0) = 5c_2 + F_0/10 = 0 \quad \boxed{c_2 = -F_0/50}$$

Therefore $\boxed{x(t) = \left(10 - \frac{F_0}{10}\right)\cos(5t) - \frac{F_0}{50}\sin(5t) + \frac{F_0}{10}t\cos(5t)}$

As $t \to \infty$ the spring will break since $x \to \infty$

E96 Forced - Damped - oscillator

$$\ddot{x} + 5\dot{x} + 6x = \sin(t) \quad \text{with} \quad x(0) = 0 \;\&\; \dot{x}(0) = 3$$

Startegy: ① find x_h ② find x_p ③ Assemble $x = x_h + x_p$ and apply intial conditions,

① $\lambda^2 + 5\lambda + 6 = 0 \Rightarrow (\lambda+3)(\lambda+2) = 0 \;\therefore\; \lambda_1 = -2,\; \lambda = -3$

$$x_h = c_1 e^{-2t} + c_2 e^{-3t}$$

② Clearly no overlap so
$$x_p = A\sin t + B\cos t$$
$$\dot{x}_p = A\cos t - B\sin t$$
$$\ddot{x}_p = -A\sin t - B\cos t = -x_p$$

Subst. into ODEq^n:

$$\ddot{x}_p + 5\dot{x}_p + 6x_p = -x_p + 5(A\cos t - B\sin t) + 6x_p$$
$$= 5(A\sin t + B\cos t) + 5(A\cos t - B\sin t)$$
$$= (5A - 5B)\sin t + (5B + 5A)\cos t = \sin t$$

Thus comparing coefficients,

$$5A - 5B = 1 \Rightarrow A = \tfrac{1}{5}(5B+1) = B + 1/5$$
$$5B + 5A = 0 \Rightarrow A = -B$$

That is $B + 1/5 = -B \Rightarrow 2B = -1/5 \;\therefore\; \boxed{B = -1/10 \;\&\; A = 1/10}$

③ Gen. Sol^n: $x(t) = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{10}(\sin t - \cos t)$

$$\dot{x}(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t} + \tfrac{1}{10}(\cos t + \sin t)$$

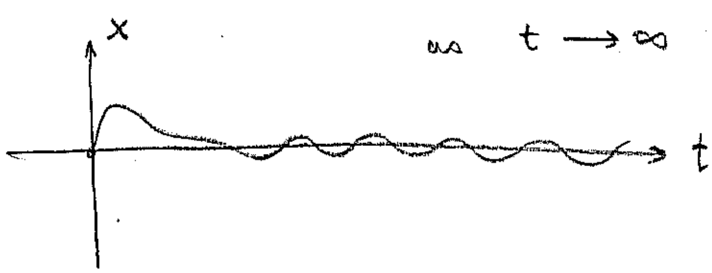
$$x(0) = c_1 + c_2 - \tfrac{1}{10} = 0 \Rightarrow c_1 = \tfrac{1}{10} - c_2$$

$$\dot{x}(0) = -2c_1 - 3c_2 + \tfrac{1}{10} = 3 \Rightarrow \underbrace{-2\left(\tfrac{1}{10} - c_2\right) - 3c_2 + \tfrac{1}{10} = 3}$$

$$-c_2 = \tfrac{30}{10} - \tfrac{1}{10} + \tfrac{2}{10} = \tfrac{31}{10} \;\therefore\; c_2 = \tfrac{-31}{10}$$

$$c_1 = \tfrac{1}{10} + \tfrac{31}{10} = \tfrac{32}{10} = c_1$$

$$\boxed{x(t) = 3.2e^{-2t} - 3.1e^{-3t} + 0.1(\sin(t) - \cos(t))}$$



as $t \to \infty$ $\quad x \to x_p \leftarrow$ the steady state sol^n

General sol^{n} to forced underdamped harmonic oscillator:

I wish to solve $m\ddot{x} + b\dot{x} + kx = F_o \cos\gamma t$ where m, b, k, F_o, γ are constants and $m, b, k > 0$ such that $b^2 < 4mk$. This means $m\lambda^2 + b\lambda + k = 0$ has only complex roots $\lambda = \frac{-b \pm i\sqrt{4mk - b^2}}{2m}$

It follows the homogeneous sol^{n} has the form:

$$x_h(t) = A e^{\frac{-bt}{2m}} \sin\left[\left(\frac{\sqrt{4mk-b^2}}{2m}\right)t + \phi\right]$$

The forcing function $F_o \cos\gamma t$ does not overlap this homogeneous sol^{n} so $x_p = B\cos\gamma t + C\sin\gamma t$ will suffice for a particular sol^{n},

$$\dot{x}_p = -B\gamma\sin\gamma t + C\gamma\cos\gamma t$$

$$\ddot{x}_p = -\gamma^2 x_p$$

Substitute into Newton's 2nd Law,

$$m\ddot{x}_p + b\dot{x}_p + kx_p = F_o \cos\gamma t$$

$$(-m\gamma^2 + k)(B\cos\gamma t + C\sin\gamma t) + b(-B\gamma\sin\gamma t + C\gamma\cos\gamma t) = F_o\cos\gamma t$$

$$\cos\gamma t\left[B(k-m\gamma^2) + bC\gamma\right] + \sin\gamma t\left[C(k-m\gamma^2) - bB\gamma\right] = F_o\cos\gamma t$$

Equate coefficients of sine and cosine yields,

$$F_o = B(k-m\gamma^2) + bC\gamma$$

$$0 = C(k-m\gamma^2) - bB\gamma \longrightarrow B = \left(\frac{k-m\gamma^2}{b\gamma}\right)C$$

Substitute into 1st eq^{n},

$$F_o = \frac{(k-m\gamma^2)^2}{b\gamma}C + b\gamma C \Rightarrow C = \frac{b\gamma F_o}{(k-m\gamma^2)^2 + (b\gamma)^2} \Rightarrow B = \frac{(k-m\gamma^2)F_o}{(k-m\gamma^2)^2 + b^2\gamma^2}$$

Thus,

$$x_p(t) = \frac{F_o}{(k-m\gamma^2)^2 + b^2\gamma^2}\left[(k-m\gamma^2)\cos\gamma t + b\gamma\sin\gamma t\right]$$

For the purpose of analyzing graph of $x_p(t)$ it is best to rewrite the sol^{n} in terms of a single sine function.
Follows same algebra as on 103 except here $x_p = A_F \sin(\gamma t + \theta)$

$$C_1 = \frac{F_o(k-m\gamma^2)}{(k-m\gamma^2)^2 + b^2\gamma^2} \quad \text{and} \quad C_2 = \frac{F_o b\gamma}{(k-m\gamma^2)^2 + b^2\gamma^2}$$

Thus, $\tan\theta = \frac{C_1}{C_2} = \frac{k-m\gamma^2}{b\gamma}$ and (continued ⤵)

Continuing,

$$A_F = \sqrt{C_1^2 + C_2^2}$$

$$= \sqrt{\frac{F_0^2 (k - m\gamma^2)^2 + F_0^2 (b\gamma)^2}{[(k - m\gamma^2)^2 + b^2\gamma^2]^2}}$$

$$= F_0 \sqrt{\frac{(k - m\gamma^2)^2 + b^2\gamma^2}{[(k - m\gamma^2)^2 + b^2\gamma^2]^2}}$$

$$= \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}$$

Therefore, we have the general solⁿ of $m\ddot{x} + b\dot{x} + kx = F_0 \cos \gamma t$ in the case $b^2 < 4mk$ of : (recall θ given by $\tan\theta = (k - m\gamma^2)/b\gamma$

$$\boxed{x(t) = \underbrace{A e^{-bt/2m} \sin\left(\frac{\sqrt{4mk - b^2}}{2m} t + \phi\right)}_{\text{transient part of sol}^{\text{n}}\text{, due to initial conditions, eventually becomes neglible}} + \underbrace{\frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta)}_{\text{steady-state sol}^{\text{n}}}}$$

transient part of solⁿ, due to initial conditions, eventually becomes neglible

steady-state solⁿ. Effective amplitude is

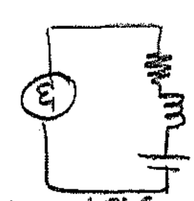
$$\frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} = MF_0$$

COMMENT: the steady-state solⁿ is largest when the so-called "gain factor" M is largest. If we have control over γ we can select a forcing function with optimal frequency $\gamma/2\pi$. Notice γ is an "angular frequency", to convert to frequency we simply divide by 2π. The frequency which <u>maximizes</u> M is called the <u>resonant</u> frequency and it can be shown that (see 243-244, it's just calculus I)

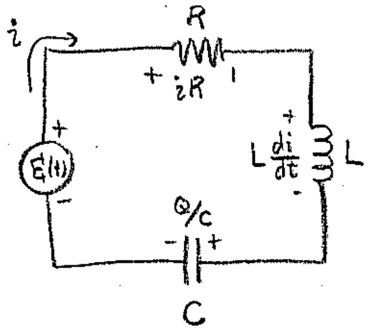
$$\boxed{f_r = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}}$$

where $\gamma_r = 2\pi f_r$ and <u>f_r = resonant frequency</u>

<u>Remark</u>: the same analysis applies to an RLC circuit with $\mathcal{E}(t) = \mathcal{E}_0 \cos \gamma t$. There will be an source voltage frequency which <u>maximizes</u> the long-term voltage in the circuit. See 108 for a dictionary between springs & RLC math.



RLC : Circuits : §5.7



Kirchoff's Law: energy is conserved.

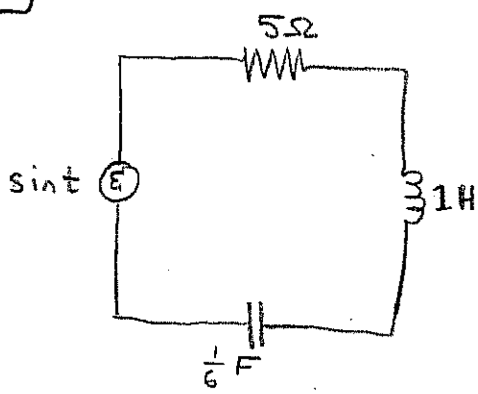
$$iR + L\frac{di}{dt} + \frac{Q}{c} = \mathcal{E}(t)$$

Where $i = \frac{dQ}{dt}$ the rate of charge passing thru the circuit.

So we are again faced with a non-homogeneous 2nd order ODE:

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{c}Q = \mathcal{E}(t)}$$

E97 Find the current for the following circuit if $Q(0) = 0$, $i(0) = 3$,



$$\underbrace{\frac{d^2Q}{dt^2} + 5\frac{dQ}{dt} + 6Q = \sin(t)}_{\text{Same math as } \boxed{E3}\,!}$$

$$\boxed{Q(t) = 3.2e^{-2t} - 3.1e^{-3t} + 0.1(\sin t - \cos t)}$$

But remember we want $i(t) = \frac{dQ}{dt}$ so differentiate:

$$\boxed{i(t) = -6.4e^{-2t} + 9.3e^{-3t} + 0.1(\cos t + \sin t)}$$

The steady state sol$^{\underline{n}}$ is $i_p = 0.1(\cos t + \sin t)$ (as $t \to \infty$)

Comparison of RLC & Damped-Forced Oscillators

Springs		RLC Circuits	
x	displacement	Q	charge
$\dot{x}$	velocity	$i = \dot{Q}$	current
m	mass	L	inductance
b	damping const.	R	resistance
k	spring const.	$1/c$	recip. of capacitance
$F(t)$	Forcing Force	$\mathcal{E}(t)$	Source Voltage

Coulombs = C

$\frac{\text{Coulombs}}{\text{second}}$ = Ampere =

Henrys = H

Ohms = Ω

{Capacitance} = F = Farad

Voltage = $\frac{\text{Pot. Energy}}{\text{unit charge}}$

$V = Q/C \rightarrow \text{Volts} = \frac{\text{Coulombs}}{\text{Farads}}$

$V = \frac{J}{C}$