Inverse Laplace Transforms

Def^{n}/ Given $F(s)$, if there is a function $f(t)$ continous on $[0,\infty)$ wi

$$\mathcal{L}\{f\} = F$$

then we say $f(t)$ is the <u>inverse Laplace transform of F</u>. We denote $f = \mathcal{L}^{-1}\{F\}$.

<u>Remark</u>: there are many possible choices for f given some particular F. This is due to fact that $\int_0^\infty e^{-st} f_1(t)\,dt = \int_0^\infty e^{-st} f_2(t)\,dt$ provided $f_1(t)$ & $f_2(t)$ disagree only at a few points. The result of the inverse transform is unique if we require f to be continuous. This is a subtle point & I've already said to much here.

E119 $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t) = \boxed{t^2}$ since $\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}$.

E120 $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}(t) = \boxed{\sin(3t)}$ since $\mathcal{L}\{\sin(3t)\}(s) = \frac{3}{s^2+9}$

E121 $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}(t)$ ← Completed square in the denominator.

$$\therefore \quad \boxed{f(t) = e^t\cos(2t)} \quad \text{as} \quad \mathcal{L}\{f\}(s) = \frac{s-1}{(s-1)^2+4}.$$

<u>Reminder</u>: to complete the square we simply want to <u>rewrite</u> a quadratic from $x^2+bx+c \longrightarrow (x-h)^2+k$. To do this we just take $\frac{1}{2}$ of coefficient of x and then $\left(x+\frac{b}{2}\right)^2 = x^2+bx+\frac{b^2}{4}$ so we then have to subtract $b^2/4$ to be fair,

$$x^2+bx+c = \left(x+\frac{b}{2}\right)^2 - \frac{b^2}{4} + c.$$

It's easier for specific examples, as it stands we are getting dangerously close to deriving the quadratic formula.

$$x^2+2x+5 = (x+1)^2-1+5 = (x+1)^2+4$$

$$x^2+6x+5 = (x+3)^2-9+5 = (x+3)^2-4$$

In practice I just make sure the LHS & RHS are equal, you don't need to remember some algorithm really.

Th$^{\underline{m}}$(7) Inverse Laplace Transform is Linear provided we choose continous $f(t)$

$$\mathcal{L}^{-1}\{F+G\} = \mathcal{L}^{-1}\{F\} + \mathcal{L}^{-1}\{G\}$$
$$\mathcal{L}^{-1}\{cF\} = c\,\mathcal{L}^{-1}\{F\}$$

__Proof:__ follows from Linearity of $\mathcal{L}$.

$$\mathcal{L}\{\mathcal{L}^{-1}\{F\} + \mathcal{L}^{-1}\{G\}\} = \mathcal{L}\{\mathcal{L}^{-1}\{F\}\} + \mathcal{L}\{\mathcal{L}^{-1}\{G\}\}$$
$$= F + G$$

Then $\mathcal{L}^{-1}\{F\} + \mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\{F+G\}$, using $\mathcal{L}^{-1}\mathcal{L} = 1$ which is true as we choose $\mathcal{L}^{-1}\{F\}$ to be continous. The fact $c \in \mathbb{R}$ pulls out follows similarly.

E122 Find $f(t) = \mathcal{L}^{-1}\{F\}(t)$ for $F(s) = \dfrac{3s+2}{s^2+2s+10}$

notice $s^2+2s+10 = (s+1)^2+9 \Rightarrow e^{-t}\cos 3t$ & $e^{-t}\sin 3t$ in the answer. Lets work it out,

$$\frac{3s+2}{s^2+2s+10} = \frac{3s+2}{(s+1)^2+9} \quad : \text{completing square}$$
$$= \frac{3(s+1)}{(s+1)^2+9} + \frac{-3+2}{(s+1)^2+9}$$
$$= 3\frac{s+1}{(s+1)^2+3^2} - \frac{1}{3}\cdot\frac{3}{(s+1)^2+3^2}$$

want to make function of $(s+1)$ then I had to subtract 3 since I added 3.

In view of the algebra above it should be obvious that

$$\mathcal{L}^{-1}\{F\}(t) = \boxed{3e^{-t}\cos(3t) - \frac{1}{3}e^{-t}\sin(3t) = f(t)}$$

E123 Consider $F(s) = \dfrac{s}{s^2+5s+6}$ find $\mathcal{L}^{-1}\{F\}(t) = f(t)$

Notice $\dfrac{s}{s^2+5s+6} = \dfrac{s}{(s+3)(s+2)} = \dfrac{A}{s+3} + \dfrac{B}{s+2}$ (recall __partial fractions__)

$$\Rightarrow s = A(s+2) + B(s+3) \quad \xrightarrow{s=-2} \; -2 = B \longrightarrow B=-2$$
$$\xrightarrow{s=-3} \; -3 = -A \longrightarrow A=3$$

Thus we deduce

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\frac{3}{s+3} - \frac{2}{s+2}\right\}(t)$$
$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t) - 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) = \boxed{3e^{-3t} - 2e^{-2t} = f(t)}$$

PARTIAL FRACTIONS

So we have discussed how polynomials split into linear and irred. quadratic factors. This means if we have a rational function which is $\frac{p(s)}{q(s)}$ then $p(s)$ & $q(s)$ will factor, we assume $\deg(p) < \deg(q)$ for convenience (otherwise we'd do long division). In short, partial fractions says you can split up a rational function into a sum of "basic" rational functions. For "basic" rational functions we can readily see how to take the inverse transform. Partial fractions involves a number of cases as you may read in the text, but it is important to realize it is nothing more than undoing making a common denominator. I'll leave you with a few examples,

$$\frac{s^3-3}{(s+1)^3(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{Ds+E}{s^2+1}$$

$$\frac{s+3}{s^2(s-2)(s^2+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{Ds+E}{s^2+3} + \frac{Fs+G}{(s^2+3)^2}\ *$$

It is a simple, but tedious, matter to calculate the constants $A, B, C, \ldots, G$ in the above. Notice on the RHS almost all the terms are easily inverse transformed. The term $*$ is subtle, just like in integration theory.

<u>Remark</u>: it is <u>crucial</u> to understand the difference between (s^2+1) and $(s+1)^2$.
Note that the inverse transform of
$\left(\frac{1}{s^2+1}\right)$ verses $\left(\frac{1}{(s+1)^2}\right)$ is quite different.

E124

$$F(s) = \frac{s+1}{s^2-2s+5} = \frac{s+1}{(s-1)^2+4} = \frac{(s-1)+2}{(s-1)^2+4}$$

$$\therefore \quad \mathcal{L}^{-1}\{F\}(t) = \boxed{e^t\cos(2t) + e^t\sin(2t)}$$

E125

$$F(s) = \frac{7s^3-2s^2-3s+6}{s^3(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2}$$

$$7s^3-2s^2-3s+6 = As^2(s-2) + Bs(s-2) + C(s-2) + Ds^3$$

$\underline{s=0|}$ $\quad 6 = -2C \quad \therefore \; C = -3$

$\underline{s=1|}$ $\quad 8 = -A - B + 3 + D \quad (*)$

$\underline{s=2|}$ $\quad 56-8-\cancel{6+6} = 8D \Rightarrow 48 = 8D \quad \therefore \; D = 6$

$\underline{s=-1|}$ $\quad -\cancel{7}-\cancel{2}+3+\cancel{6} = A(-3) - B(-3) - 3(-3) - \cancel{6}$

$$0 = -3A + 3B + 3$$

$$\Rightarrow A = B + 1$$

$(*)$ becomes $\quad 8 = -B - 1 - B + 3 + 6$

$\quad 0 = -2B \quad \therefore \; B = 0 \qquad \therefore \; A = 1$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{3}{s^3} + \frac{6}{s-2}\right\} \qquad \text{note } \mathcal{L}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$$

$$= \boxed{1 - \frac{3}{2}t^2 + 6e^{2t}}$$

E126 Calculate the inverse transform of F given below:

$$F(s) = e^{-s}\left(\frac{4s+2}{s(s+1)}\right)$$

$$\frac{4s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$4s+2 = A(s+1) + Bs$$

s| $4 = A + B$

Const.| $2 = A \Rightarrow B = 2$

$$G(s) = \frac{2}{s} + \frac{2}{s+1} \quad \therefore \quad g(t) = 2 + 2e^{-t}$$

$$\mathcal{L}^{-1}\{G(s)e^{-s}\} = g(t-1)\,u(t-1)$$ Thm (8) eq. 6

$$= \left(2 + 2e^{-(t-1)}\right)u(t-1)$$

$$= \boxed{2\left(1 + e^{-t+1}\right)u(t-1)}$$