USING LAPLACE TRANSFORMS TO SOLVE IVPs.

E127 $y'' - 2y' + 5y = 0$ with $y(0) = 2$ & $y'(0) = 12$

Take the Laplace transform,

$$s^2\bar{Y} - s\,y(0) - y'(0) - 2(s\bar{Y} - y(0)) + 5\bar{Y} = 0$$

Lets solve for $\bar{Y}$,

$$(s^2 - 2s + 5)\bar{Y} = 2s + 8$$

$$\bar{Y}(s) = \frac{2s+8}{s^2-2s+5}$$

We need to determine $y(t)$, so all we really need is to find $\mathcal{L}^{-1}\{\bar{Y}\}(t)$. To do that we need to determine if $s^2 - 2s + 5$ will factor, note $b^2 - 4ac = 4 - 20 = -16 < 0$ thus it is an irreducible quad $\Rightarrow$ complete square then break into sine & cosine like pieces.

$$\frac{2s+8}{s^2-2s+5} = \frac{2s+8}{(s-1)^2+4}$$

$$= \underbrace{\frac{2(s-1)}{(s-1)^2+2^2}}_{\text{cosine}} + \underbrace{\frac{2}{2}\,\frac{8+2}{(s-1)^2+2^2}}_{\text{sine}}$$

Thus

$$\mathcal{L}^{-1}\{\bar{Y}\}(t) = \boxed{2e^t\cos(2t) + 5e^t\sin(2t) = y(t)}$$

Pictorial Summary

$$\begin{tikzcd}
\boxed{\text{Differential Eq}^{n} \text{ in } t \text{ and } y} \arrow[r, "\mathcal{L}"] \arrow[ddr, dashed, "\text{other methods}"'] & \boxed{\text{Algebraic Eq}^{n} \text{ in } s \text{ and } \bar{Y}} \arrow[d, "\text{Solve for } \bar{Y}\text{, partial fractions}"] \\
& \boxed{\bar{Y} = \text{stuff}} \arrow[d, "\mathcal{L}^{-1}"] \\
& \boxed{y(t) = \text{general Sol}^{n}}
\end{tikzcd}$$

- notice this method automatically encodes the intial conditions.

E128 Solve the repeated root problem $y'' + 4y' + 4y = 0$ subject to the initial conditions $y(0) = 1$ and $y'(0) = 1$. Taking Laplace transform yields,

$$s^2 Y - s\,y(0) - y'(0) + 4(sY - y(0)) + 4Y = 0$$

$$(s^2 + 4s + 4)Y = s + 5$$

$$Y = \frac{s+5}{s^2+4s+4} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

$$s + 5 = A(s+2) + B$$

$$\underline{\text{Const}} \quad 5 = 2A + B$$

$$\underline{s} \quad \underline{1 = A} \quad \Rightarrow \quad B = 5 - 2A = \underline{3 = B}$$

$$\therefore \quad y(t) = \mathcal{L}^{-1}\{Y\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2}\right\}(t)$$

$$\Rightarrow \quad \boxed{y(t) = e^{-2t} + 3te^{-2t}}$$

<u>Remark</u>: The method of Laplace transforms has derived the curious te^{-2t} term. Before we just pulled it out of thin-air and argued that it worked. In defense of our earlier methods, the Laplace machine is not that intuitive either. At least we have one derivation now. Another route to explain the "t" in the double root sol^n is to use "reduction of order" (see 97). There is also a pretty derivation based on the matrix exponential and generalized eigenvectors, but we'll not go over systems in this course. We may get to it in math 321.

E129

$$y'' + y = 2e^{t} \qquad y(0) = 1 \, , \; y'(0) = 2$$

$$s^2 Y - s - 2 + Y = \frac{2}{s-1}$$

$$(s^2+1) Y = s + 2 + \frac{2}{s-1}$$

$$Y = \frac{s+2}{s^2+1} + \underline{\frac{2}{(s^2+1)(s-1)}}$$

nontrivial, need to do partial fractions to break it up.

$$\frac{2}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

$$2 = (As+B)(s-1) + C(s^2+1)$$

$$2 = s^2(A+C) + s(B-A) + C - B$$

s^2 | $0 = A + C \longrightarrow \underline{A = -C}$

s | $0 = B - A \longrightarrow A = B = -C$

const | $2 = C - B \longrightarrow 2 = C - (-C) = 2C \; \therefore \; C = 1 \Rightarrow A = B = -1$

Therefore,

$$Y = \cancel{\frac{s}{s^2+1}} + \frac{2}{s^2+1} + \underbrace{\left(\frac{\cancel{-1s} - 1}{s^2+1}\right) + \frac{1}{s-1}}_{\text{from the par. frac.}}$$

$$= \frac{1}{s^2+1} + \frac{1}{s-1}$$

Taking $\mathcal{L}^{-1}$ yields

$$\therefore \; \boxed{y = \sin t + e^{t}}$$

E130 Solve $w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}$ given $w(\pi)=2$, $w'(\pi)=12$.
We need conditions at $t=0$ so to remedy being given them at π we introduce

$$Y(t) \equiv W(t+\pi) \Rightarrow \begin{array}{l} Y(0) = W(\pi) = 2 \\ Y'(0) = W'(\pi) = 12 \end{array}$$

Then,

$$w''(t+\pi) - 2w'(t+\pi) + 5w(t+\pi) = -8e^{\pi-(t+\pi)} = -8e^{-t}$$

Thus

$$Y'' - 2Y' + 5Y = -8e^{-t} \quad \text{with } Y(0)=2 \text{ \& } Y'(0)=12.$$

Taking Laplace transform yields

$$(s^2-2s+5)\mathcal{Y} - 2s - 12 - 2(-2) = \frac{-8}{s+1}$$

$$\mathcal{Y} = \left(8+2s - \frac{8}{s+1}\right)\frac{1}{s^2-2s+5}$$

$$= \frac{3(s-1)+2(4)}{(s-1)^2+2^2} - \frac{1}{s+1}$$

partial fractions ½ page of algebra here.

$$\mathcal{L}^{-1}\{\mathcal{Y}\}(t) = \underline{3e^{t}\cos(2t) + 4e^{t}\sin(2t) - e^{-t} = Y(t)}$$

Then returning to our original problem note $W(t) = Y(t-\pi)$.

$$W(t) = 3e^{t-\pi}\cos(2(t-\pi)) + 4e^{t-\pi}\sin(2(t-\pi)) - e^{-t+\pi}$$

$$= 3e^{t-\pi}\cos(2t-2\pi) + 4e^{t-\pi}\sin(2t-2\pi) - e^{\pi-t}$$

$$= \boxed{3e^{t-\pi}\cos(2t) + 4e^{t-\pi}\sin(2t) - e^{\pi-t} = W(t)}$$

<u>Remark</u>: this example is important in that it show us how to use Laplace transforms to treat problems where the data is given at any time, not just zero as the formalism is set-up for.

$\boxed{E131}$ (shifted initial conditions) Suppose $w(1)=1$ and $w'(1)=0$. Solve

$$w''(t) - w(t) = \sin(t-1)$$

Using Laplace transforms. Introduce $y(t) = w(t+1)$ so that $y(0) = w(1)$. Also notice that
$y'(t) = \frac{dw}{dt}\frac{d}{dt}(t+1) = w'(t+1)$ and $y''(t) = w''(t+1)$
Consider the differential equation at $t+1$,

$$w''(t+1) - w(t+1) = \sin(t+1-1)$$

Thus, $y''(t) - y(t) = \sin(t)$, and $y(0)=1$, $y'(0)=0$
Now we can use the standard Laplace theory on y,

$$s^2 Y - s - Y = \frac{1}{s^2+1}$$

$$Y = \frac{1}{s^2-1}\left(s + \frac{1}{s^2+1}\right)$$

$$= \frac{s(s^2+1)+1}{(s+1)(s-1)(s^2+1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C+Ds}{s^2+1}$$

$$\Rightarrow \quad s(s^2+1)+1 = A(s-1)(s^2+1) + B(s+1)(s^2+1) + (s^2-1)(C+Ds)$$

$$\Rightarrow \quad s^3+s+1 = A(s^3-s^2+s-1) + B(s^3+s^2+s+1) + C(s^2-1) + D(s^3-s)$$

Equating Coefficients of $s^3, s^2, s, 1$ yields,

$\underline{s^3}$ $1 = A + B + D$

$\underline{s^2}$ $0 = -A + B + C$

$\underline{s}$ $1 = A + B - D$

$\underline{1}$ $1 = -A + B - C$

Well that's not very fun, notice $s=1$ gives $3 = 4B$
whereas $s=-1$ gives $-1 = -4A$

Just found $A = 1/4$ and $B = 3/4$ so using the eq^{n}'s from last page,

$$1 = A + B + D = 1 + D \therefore \underline{D = 0}$$
$$0 = -A + B + C = \tfrac{1}{2} + C \therefore \underline{C = -1/2}$$

Thus we find that

$$Y = \frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{3}{4}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

Thus $y(t) = \mathcal{L}^{-1}\{Y\}(t)$ is clearly,

$$y(t) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{t} - \frac{1}{2}\sin(t)$$

Now to finish the problem we convert back to w using the fact $w(t) = y(t-1)$ thus,

$$\boxed{w(t) = \frac{1}{4}e^{-(t-1)} + \frac{3}{4}e^{t-1} - \frac{1}{2}\sin(t-1)}$$

There are other perhaps simpler ways to express our final answer, but this will suffice.

§7.5 #36 $\quad tY'' - tY' + Y = 2 \qquad Y(0) = 2, \quad Y'(0) = -1$

$$\mathcal{L}\{tY''\} = -\frac{d}{ds}\mathcal{L}\{Y''\}$$
$$= -\frac{d}{ds}\left[s^2\bar{Y} - 2s + 1\right]$$
$$= -2s\bar{Y} - s^2\bar{Y}'(s) + 2$$

$$\mathcal{L}\{tY'\} = -\frac{d}{ds}\left[\mathcal{L}\{Y'\}\right]$$
$$= -\frac{d}{ds}\left[s\bar{Y} - 2\right]$$
$$= -\bar{Y} - s\bar{Y}'(s)$$

Thus

$$\mathcal{L}\{tY'' - tY' + Y\} = \mathcal{L}\{2\}$$
$$-2s\bar{Y} - s^2\bar{Y}'(s) + 2 + \bar{Y} + s\bar{Y}'(s) + \bar{Y} = \frac{2}{s}$$

$$\frac{d\bar{Y}}{ds}\left(s - s^2\right) + \bar{Y}\left(2 - 2s\right) = \frac{2}{s} - 2$$

$$-\frac{d\bar{Y}}{ds}s(s-1) - 2\cancel{s}(s-1)\bar{Y} = 2\left(\frac{1}{s} - 1\right)$$

$$\frac{d\bar{Y}}{ds}s(\cancel{s-1}) + 2(\cancel{s-1})\bar{Y} = 2\left(1 - \frac{1}{s}\right) = 2\left(\frac{\cancel{s-1}}{s}\right)$$

$$s\frac{d\bar{Y}}{ds} + 2\bar{Y} = \frac{2}{s}$$

$$\frac{d\bar{Y}}{ds} + \underbrace{\frac{2}{s}}_{P}\bar{Y} = \underbrace{\frac{2}{s^2}}_{Q} \qquad \mu = \exp\int\frac{2}{s}ds = \exp(\ln(s^2)) = s^2$$

integrating factor! method.

$$\bar{Y} = \frac{1}{s^2}\int \cancel{s^2}\frac{2}{\cancel{s^2}}ds = \frac{1}{s^2}\cdot(2s + c_1) = \frac{2}{s} + \frac{c_1}{s^2}$$

$$\bar{Y}(s) = \frac{2}{s} \Rightarrow Y(t) = 2 + c_1 t$$

$$Y'(0) = -1 \Rightarrow Y'(0) = c_1 = -1$$

$$\therefore \boxed{Y(t) = 2 - t}$$