LAPLACE TRANSFORMS OF DISCONTINUOUS & PERIODIC FUNCTIONS

One main motivation for including Laplace Transforms in your education is that it allows us to treat problems with piecewise continuous forcing terms in a systematic fashion. It would be more achward using just the standard analysis w/o Laplace. (see Problem 5 of Problem Set I)

Def^n/ The unit step function $u(t)$ is defined by

$$u(t) \equiv \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

• Curious, the value at $t=0$ is undefined. Some other texts use $\leq$ or $\geq$ anyway it doesn't matter.

Often it will be convenient to use $u(t-a) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases}$. This allows us to switch functions on and/or off for particular ranges of t

E133 $$g(t) = \begin{cases} 0 & t < 0 \\ \cos t & 0 < t < 1 \\ \sin t & 1 < t < \pi \\ t^2 & \pi < t \end{cases}$$

$$g(t) = [\underbrace{u(t)}_{\substack{\text{turns on} \\ \text{cosine at} \\ t=0}} - \underbrace{u(t-1)}_{\substack{\text{turns off} \\ \text{cosine} \\ \text{at } t=1}}]\cos t + [\underbrace{u(t-1)}_{\substack{\text{turns on} \\ \text{sine} \\ \text{at } t=1}} - \underbrace{u(t-\pi)}_{\substack{\text{turns off} \\ \text{sine} \\ \text{at } t=\pi}}]\sin t + \underbrace{u(t-\pi)}_{\substack{\text{turns on} \\ t^2 \\ \text{at } t=\pi.}} t^2$$

• It's not hard to see why this function is useful to a myriad of applications, anywhere you have a switch the unit-step function provides an idealized model of that.

Proposition (4) $\mathcal{L}\{u(t-a)\}(s) = \frac{1}{s}e^{-as}$

Proof:

$$\mathcal{L}\{u(t-a)\}(s) \equiv \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_a^\infty e^{-st} dt$$

$$= \frac{-1}{s} e^{-st}\Big|_a^\infty = \frac{-1}{s}\Big(\underbrace{e^{-s\infty}}_{\substack{0 \\ (s>0)}} - e^{-as}\Big) = \boxed{\frac{1}{s}e^{-as}}.$$

$\boxed{\begin{array}{l} Th^{m}(8) \text{ Let } F(s) = \mathcal{L}\{f\}(s) \text{ for } s > \alpha \geq 0. \text{ If } a > 0 \text{ then} \\ \quad \mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s) \\ \text{And conversely,} \\ \quad \mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a) \end{array}}$

Proof: We calculate from the definition,

$$\begin{aligned} \mathcal{L}\{f(t-a)u(t-a)\}(s) &= \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt \\ &= \int_a^\infty e^{-st} f(t-a)\,dt \\ &= \int_0^\infty e^{-s(u+a)} f(u)\,du \\ &= e^{-sa}\int_0^\infty e^{-su} f(u)\,du \\ &= e^{-sa}\,\mathcal{L}\{f\}(s) \\ &= e^{-as}F(s). \end{aligned}$$

u-substitution

$$\boxed{\begin{array}{l} u = t-a \\ u(a) = a-a = 0 \\ du = dt \\ t = u+a \end{array}}$$

must change bounds! of course $u(\infty) = \infty$ as well.

$$\boxed{\text{Corollary (8)}: \ \mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)}$$

Proof:

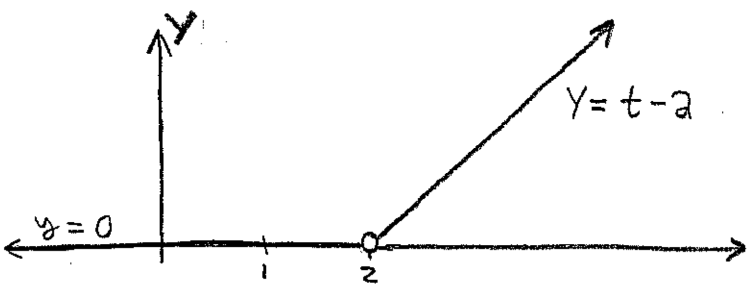
$$\begin{aligned} \mathcal{L}\{g(t)u(t-a)\}(s) &= \mathcal{L}\{h(t-a)u(t-a)\}(s) \ ; \ h(t-a) \equiv g(t). \\ &= e^{-as}\mathcal{L}\{h\}(s) \ , \text{ using } Th^{m}(8). \\ &= e^{-as}\mathcal{L}\{g(t+a)\}(s). \ , \text{ as } h(t) = g(t+a \end{aligned}$$

E134 Simply apply Cor. (8) to obtain,

$$\begin{aligned} \mathcal{L}\{t^2 u(t-1)\}(s) &= e^{-s}\mathcal{L}\{(t+1)^2\}(s) \\ &= e^{-s}\mathcal{L}\{t^2+2t+1\}(s) \\ &= \boxed{e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)} \end{aligned}$$

$\boxed{E135}$ Determine $\mathcal{L}^{-1}\left\{\frac{1}{s^2}e^{-2s}\right\}$ and sketch it's graph. Lets use $Th^{\underline{m}}$ (8),

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}e^{-2s}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t-2)\,u(t-2) \longleftarrow \left(\begin{array}{l} f(t) = \mathcal{L}^{-1}\{F\}(t) \\ f(t-a) = \mathcal{L}^{-1}\{F\}(t-a) \end{array}\right.$$
$$= (t-2)\,u(t-2)$$



Remark: If we find exponential factors in the s-domain that suggest we'll encounter unit-step functions upon taking $\mathcal{L}^{-1}$ to get back to the t-domain.

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Def^{n}/ A function $f(t)$ is said to be periodic with period T if
$$f(t+T) = f(t) \quad \text{for all } t \in \text{dom}(f)$$

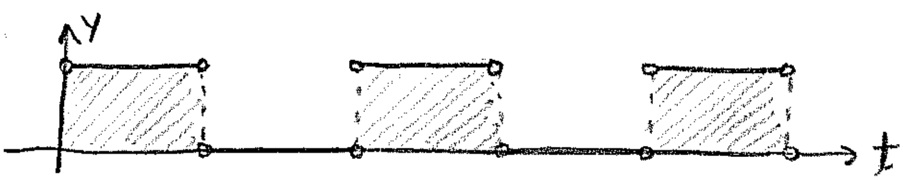
Examples

$f_1(t) = \sin t$, has $T_1 = 2\pi$

$f_2(t) = \sin(kt)$, has $kT_2 = 2\pi \Rightarrow T_2 = \frac{2\pi}{k}$.

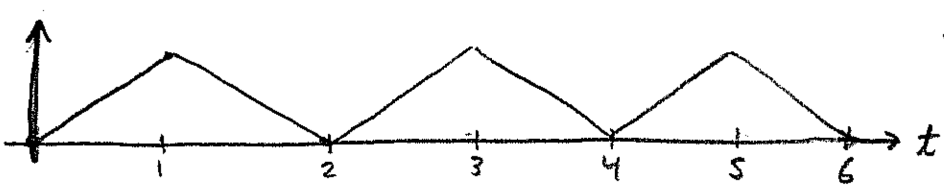
$f_3(t) = \tan(t)$, has $T_3 = \pi$

$f_4(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 < t < 2 \end{cases}$ with $T = 2$



$Y = f_4(t)$ the "square wave"

$f_5(t) = \begin{cases} t & 0 < t < 1 \\ 2-t & 1 < t < 2 \end{cases}$ with $T = 2$



$Y = f_5(t)$

Def^{n}/ For $f(t)$ with $[0,T] \subset \text{dom}(f)$ with f periodic with period T we define the "windowed version of f"
$$f_T(t) = \begin{cases} f(t) & 0 < t < T \\ 0 & \text{other } t \in \text{dom}(f) \end{cases}$$

The Laplace Transform of the windowed version of a periodic function $f(t)$ (with period T) is similarly denoted

$$F_T(s) = \int_0^{\infty} e^{-st} f_T(t)\,dt = \int_0^T e^{-st} f(t)\,dt$$

$$\boxed{\begin{array}{l} \text{Th}^{\underline{m}}\,(9)\quad \text{If } f \text{ has period } T \text{ and is piecewise continuous on } [0,T] \text{ then} \\ \qquad \mathcal{L}\{f\}(s) = \dfrac{F_T(s)}{1-e^{-sT}} = \dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} \end{array}}$$

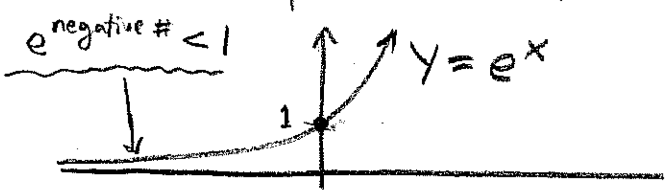
<u>Proof:</u> Use the unit step function to write, assume $\text{dom}(f) = [0,\infty)$

$$f(t) = f_T(t) + f_T(t-T)\,u(t-T) + f_T(t-2T)\,u(t-2T) + \cdots$$

This is sneaky in that $f_T(t-T) \neq 0$ only for $0 < t-T < T$ that is $T < t < 2T$ and $f_T(t-2T) \neq 0$ only for $2T < t < 3T$ so the unit step functions just multiply by 1 and are superfluous as these shifted f_T functions are already set-up to be zero most places. We want the unit step functions so we can use Th$^{\underline{m}}$ (8).

$$\begin{aligned}
\mathcal{L}\{f\}(s) &= \mathcal{L}\{f_T\}(s) + \mathcal{L}\{f_T(t-T)u(t-T)\}(s) + \mathcal{L}\{f_T(t-2T)u(t-2T)\}(s) \\
&= F_T(s) + e^{-sT}F_T(s) + e^{-2sT}F_T(s) + \cdots \\
&= F_T(s)\left[1 + e^{-sT} + (e^{-sT})^2 + (e^{-sT})^3 + \cdots\right] \\
&= a(1 + r + r^2 + r^3 + \cdots) \qquad \text{geometric series} \\
&= \frac{a}{1-r} \quad \text{for } |r| < 1 \qquad a = F_T(s),\ r = e^{-sT} \\
&= \frac{F_T(s)}{1-e^{-sT}} \quad \text{for } |e^{-sT}| < 1
\end{aligned}$$

Notice that if $s > 0$ and $T > 0$ (by assumption) then $-sT < 0 \Rightarrow e^{-sT} < 1$, Just think about the graph of the exponential function:



$\frac{d}{dx}(e^x) = e^x > 0$

e^x always increases so for $x < 0$ $e^x < e^0 = 1$.

E136 $f_T(t) = e^t$ and periodic $f(t)$ has $T = 1$. Calculate the Laplace transform of this function

$$\mathcal{L}\{f\}(s) = \frac{\int_0^1 e^{-st}e^t\,dt}{1-e^{-s}}$$

$$= \frac{1}{1-e^{-s}}\int_0^1 e^{t(1-s)}\,dt$$

$$= \frac{1}{1-e^{-s}}\,\frac{1}{1-s}\,e^{t(1-s)}\Big|_0^1$$

$$= \frac{1}{1-e^{-s}}\,\frac{1}{1-s}\left(e^{1-s}-1\right) = \boxed{\frac{1}{s-1}\left(\frac{e^s-e}{e^s-1}\right)}$$

whichever really.

E137 Let $f(t) = \begin{cases} \frac{\sin t}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$ (since $\lim_{t\to 0}\frac{\sin t}{t} = 1$ this is a continuous function for whatever thats worth.)

Anyway, recall

$$\sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots$$

$$\Rightarrow \quad \frac{\sin t}{t} = 1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^5 + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!}$$

It's not hard to see that $f(t)$ is of exponential order, thus we expect it's Laplace transform exists. And in fact it can be shown that the Laplace transform of a series is the series of the Laplace transforms of the terms. That is we can extend linearity of $\mathcal{L}$ to infinite sums provided the series is well behaved (need exponential order)

$$\mathcal{L}\{f\}(s) = \mathcal{L}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!}\right\}(s)$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\,\mathcal{L}\{t^{2n}\}(s)$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\,\frac{(2n)!}{s^{2n+1}}, \quad \text{but } \frac{(2n)!}{(2n+1)!} = \frac{1}{2n+1}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\,\frac{1}{s^{2n+1}} \underset{(*)}{=} \tan^{-1}\left(\frac{1}{s}\right).$$

(*) Since $\tan^{-1}(x) = \int\frac{d}{dx}\left(\tan^{-1}(x)\right)dx = \int\frac{1}{1+x^2}\,dx = \int(1-x^2+x^4-x^6+\cdots)\,dx$

Calc. II arguments $\quad = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$

$$\therefore\ \tan^{-1}\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\,\frac{1}{s^{2n+1}}$$

Def^{n}/ The gamma function $\Gamma(t)$ is defined for $t>0$ as

$$\Gamma(t) \equiv \int_0^{\infty} e^{-u} u^{t-1}\, du$$

• Property of Γ is that $\Gamma(t+1) = t\,\Gamma(t)$. Since,

$$\Gamma(t+1) = \int_0^{\infty} e^{-u} u^{t+1-1}\, du$$

I.B.P.

$\tilde{u} = u^t$	$d\tilde{v} = e^{-u} du$
$d\tilde{u} = t u^{t-1} du$	$\tilde{v} = -e^{-u}$

$$= \lim_{N\to\infty} \left(\int_0^N e^{-u} u^t\, du \right)$$

$$= \lim_{N\to\infty} \left(\tilde{u}\tilde{v}\Big|_0^N - \int_0^N \tilde{v}\, d\tilde{u} \right)$$

$$= \lim_{N\to\infty} \left(-u^t e^{-u}\Big|_0^N + \int_0^N e^{-u} t u^{t-1}\, du \right)$$

$$= \lim_{N\to\infty} \left(-\underbrace{N^t e^{-N}}_{0} + 0 + t\int_0^N e^{-u} u^{t-1}\, du \right)$$

$0 \leftarrow$ repeated application of L-Hopital's Rule.

$$= t\int_0^{\infty} e^{-u} u^{t-1}\, du$$

$$= t\,\Gamma(t)$$

Notice also that $\Gamma(1) = \int_0^{\infty} e^{-u} du = -e^{-u}\Big|_0^{\infty} = -e^{-\infty}+1 = 1$.
And for $n \in \mathbb{Z}$ we have $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$
$= n(n-1)\cdots 2\,\Gamma(1) \Rightarrow \boxed{\Gamma(n+1) = n!}$. This means the gamma function is a continous extension of the factorial ! $\leftarrow$ (the text did it 1st, I'm sorry its horrible.)

Previously, we have utilized $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$ but what do we do if $n \notin \mathbb{N} = \{1, 2, 3, \dots\}$, we use the gamma-function.

$$\boxed{\mathcal{L}\{t^n\}(s) = \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{for } n \geq 0}$$

Here n can be <u>any</u> nonnegative real #. Lets prove it, take $s > 0$ as usual,

$$\begin{aligned}
\mathcal{L}\{t^n\}(s) &= \int_0^\infty e^{-st} t^n \, dt \\
&= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s} \\
&= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n \, du \\
&= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^{n+1-1} \, du \\
&= \frac{\Gamma(n+1)}{s^{n+1}}.
\end{aligned}$$

$u = st$
$du = s\,dt$
$t^n = \left(\frac{u}{s}\right)^n$

<u>Remark</u>: the Γ-function is important to probability theory.

$\boxed{\text{E138}}$ $\mathcal{L}\{t^{3.6}\}(s) = \frac{\Gamma(4.6)}{s^{4.6}}$

Here $\Gamma(4.6)$ is some constant with $\Gamma(4) < \Gamma(4.6) < \Gamma(5)$ hence $3! < \Gamma(4.6) < 4!$ or $6 < \Gamma(4.6) < 24$. Sadly my calculator just ran out of batteries so I'll leave it at that. You can approximate

$$\Gamma(4.6) = \int_0^\infty e^{-u} u^{3.6} \, du$$

by the numerical integrator in most good calculators. Replace ∞ with 100 it'll probably work. Tinker a bit to be safe.