SEPARATION OF VARIABLES (§2.2) & APPLICATIONS

Certain 1st order ode's can be seen to have the form

$$\frac{dY}{dx} = g(x)\, f(Y)$$

In which case we can seperate the variables

$$\frac{dY}{f(Y)} = g(x)\,dx$$

And then integrate

$$\int \frac{dY}{f(Y)} = \int g(x)\,dx$$

When you can actually do these integrals this will implicitly (and sometimes explicitly) define Y in terms of x; that means you found a sol^{n}!

Pf/ $\displaystyle \int \frac{dY}{f(Y)} = \int \frac{g(x) f(Y)\,dx}{f(Y)}$: Changing variables to x $dY = g(x) f(Y)\, dx$ by assumption

$\displaystyle = \int g(x)\,dx$ //

To summarize: the justification for separation of variables is u-substitution.

[E5] $\displaystyle \frac{dY}{dt} = kY \Rightarrow \frac{dY}{Y} = k\,dt$ then integrate

$\displaystyle \int \frac{dY}{Y} = \int k\,dt \Rightarrow \underline{\ln Y = kt + C}$ (implicit sol^{n})

Thus $Y(t) = e^{kt+C} = e^{C} e^{kt} = \boxed{Y_0 e^{kt} = Y(t)}$ (explicit sol^{n})

If $Y(0) = 3$ find the sol^{n}

$Y(0) = Y_0 e^{k(0)} = \boxed{Y_0 = 3} \Rightarrow \underline{Y(t) = 3e^{kt}}$

(This is why Y_0 is good notation here)

Remark: $k > 0$ exponential growth eq^{n}.
$k < 0$ exponential decay eq^{n}

E6 $\frac{dy}{dx} = a^{x+y}$: find sol$^{\underline{n}}$ thru sep. of variables assume $a > 0$ and $a \neq 1$.

$\int a^{-y} dy = \int a^{x} dx$: seperate then integrate.

$$\frac{-1}{\ln(a)} a^{-y} = \frac{1}{\ln(a)} a^{x} + \tilde{c}$$

$$a^{-y} = -a^{x} - \ln(a)\tilde{c} = c - a^{x}$$: want to solve for y.

$\ln(a^{-y}) = \ln(c - a^{x})$: it is crucial to insure we are taking the ln of positive quantites! that is why we moved the minus sign to the other side.

$$-y \ln(a) = \ln(c - a^{x})$$

$$\boxed{y = -\frac{\ln(c - a^{x})}{\ln(a)}}$$

Notice that c is arbitrairy and can only be specified if we supply further demands (an intial or boundary condition)

E7 $\frac{du}{d\theta} = \frac{2\theta + \sec^2\theta}{2u}$ find sol$^{\underline{n}}$ with $u(0) = -5$

$2u\, du = (2\theta + \sec^2\theta)d\theta$: seperated variables, now integrate,

$u^2 = \theta^2 + \tan\theta + C$: an implicit sol$^{\underline{n}}$.

$\therefore \; u = \pm\sqrt{\theta^2 + \tan\theta + c}$; an explicit sol$^{\underline{n}}$.

$$u(0) = \pm\sqrt{c} = -5$$

$\Rightarrow -\sqrt{c} = -5$ (Must choose negative sqrt. sol$^{\underline{n}}$)

$\Rightarrow c = 25$

$$\boxed{u(\theta) = -\sqrt{\theta^2 + \tan\theta + 25}}$$

E8 $\dfrac{dy}{dx} = \dfrac{6x^5 - 2x + 1}{\cos(y) + \sec^2(y)}$, find the general solⁿ implicitly

$$\int(\cos(y) + \sec^2(y))\,dy = \int(6x^5 - 2x + 1)\,dx$$

$$\boxed{\sin(y) + \tan(y) = x^6 - x^2 + x + c}$$ ← implicit general solⁿ

E9 $y' = x^3(1-y)$ with $y(0) = 3$, find explicit solⁿ to IVP.
Recall that $y' = \frac{dy}{dx}$ when x is the independent variable,

$$\int \frac{dy}{1-y} = \int x^3 dx \Rightarrow -\ln|1-y| = \frac{1}{4}x^4 + c_1$$

$$\Rightarrow \ln|1-y| = c_2 - x^4/4$$

$$\Rightarrow |1-y| = e^{c_2 - x^4/4}$$

$$\Rightarrow 1 - y = \pm e^{c_2} e^{-x^4/4}$$

$$\Rightarrow \boxed{y = 1 + ce^{-x^4/4}}$$ ← general solution

Now $y(0) = 3 \Rightarrow 3 = 1 + c \quad \therefore \quad c = 2 \Rightarrow \boxed{y = 1 + 2e^{-x^4/4}}$ ← solⁿ to the initial value problem

E10 x = position
$v = \frac{dx}{dt}$ = velocity
$a = \frac{dv}{dt}$ = acceleration

(look ahead to §3.4 for more physics)

Suppose that the acceleration is constant, in particular consider $a = g \in \mathbb{R}$. Find velocity as a function of time, and then position.

$$\frac{dv}{dt} = g \Rightarrow \int dv = \int g\,dt \Rightarrow \boxed{v = gt + c}$$

To find v as a function of x we could find $x(t)$ then solve for t, but we can get around finding $x(t)$ if we use the chain rule $\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx}$ but $\frac{dx}{dt} = v$ so,

$$v\frac{dv}{dx} = g \Rightarrow \int v\,dv = \int g\,dx \Rightarrow \frac{1}{2}v^2 = gx + c$$

$$\Rightarrow \boxed{v = \pm\sqrt{2gx + c}}$$

• Orthogonal Trajectories (O.T.)

Given a family of curves an O.T. is a curve which is orthogonal to each member of the family; meaning at the points of interception the tangents are orthogonal (aka perpendicular, if m-slope then $\frac{-1}{m}$ is $\perp$ line's slope)

E11 $x^2+y^2=R^2$ defines a circle for each $R>0$. Let's find the orthogonal trajectories to this family of curves, diff. implicitly

$$2x+2y\frac{dy}{dx}=0 \quad \therefore \quad \frac{dy}{dx}=\frac{-x}{y} \quad \text{(for circles)}$$

Now then the O.T must have $\frac{dy}{dx}=\frac{-1}{-x/y}=\frac{y}{x}$ hence

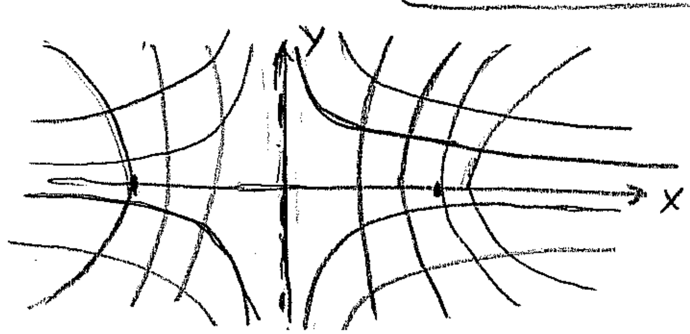
$$\frac{dy}{y}=\frac{dx}{x} \Rightarrow \ln|y|=\ln|x|+c$$
$$\Rightarrow y=e^{\ln(x)+c}=e^{\ln(x)}e^{c}=xe^{c}$$

Thus as pictured in figure 8 on pg. 516 $y=mx$ are truly the OT's to circles.

E12 $x^2-y^2=k \Rightarrow 2x-2y\frac{dy}{dx}=0 \quad \therefore \frac{dy}{dx}=\frac{-2x}{-2y}=\frac{x}{y}$

Hence the O.T. will have $\frac{dy}{dx}=\frac{-1}{x/y}=\frac{-y}{x}$

$$\therefore \frac{dy}{-y}=\frac{dx}{x} \Rightarrow -\ln|y|=\ln|x|+\tilde{c}$$
$$\Rightarrow \ln\left(\frac{1}{|y|}\right)=\ln(\bar{c}x)$$
$$\Rightarrow \frac{1}{|y|}=\bar{c}|x|$$
$$\Rightarrow \boxed{y=\frac{\pm c}{x} \text{ is the O.T.}}$$



$x^2-y^2=k$ is a hyperbola

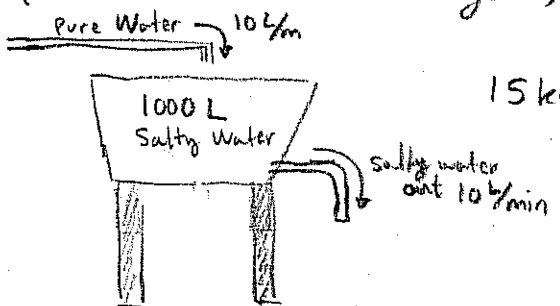
$y=0 \quad x=\pm k$

$x=0 \quad y$ d.n.e

E13 Mixing Tank Problem

Consider some tank of fixed volume with some substance entering/exiting the tank, let $Y(t)$ be the amount of the substance at time t in the tank.

(I'll work #35 for you)



15 kg of salt at $t = 0$

Let $Y(t)$ = kg of salt in tank at time t

$$\frac{dY}{dt} = (\text{rate in}) - (\text{rate out})$$
$$= 0 - \left(10 \frac{L}{\text{min}}\right) \cdot \left(\frac{Y(t)}{1000 L}\right)$$
$$= -\frac{1}{100} Y(t)/\text{min} \leftarrow \text{kg/min} \quad \text{makes sense}$$
$$= -Y/100$$

Thus $\dfrac{dY}{Y} = -\dfrac{dt}{100}$ $\therefore \ln(Y) = -\dfrac{t}{100} + \tilde{c}$ $\therefore \underline{Y(t) = Y_o e^{-t/100}}$

$Y(0) = 15\text{kg} = Y_o$ $\therefore$ $\boxed{Y(t) = 15 e^{-t/100} \text{ kg}}$

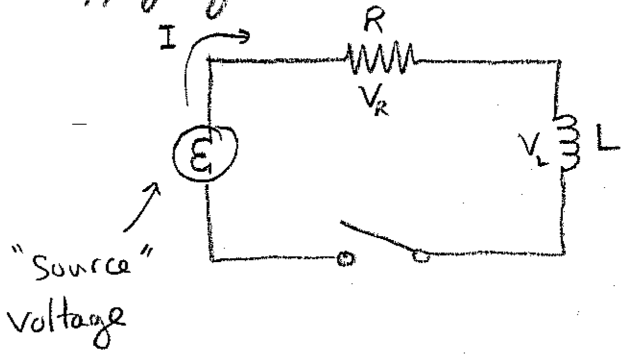
So after 20 minutes $Y(20) = 15 e^{-20/100} = \boxed{12.28 \text{ kg at } t = 20 \text{ min}}$

Remark: there are many ways to modify this problem. We could include a term to account for evaporation. We could include a term to account for the periodic addition of salt to the tank. Whatever extra influences we include the basic idea comes back to

$$\boxed{\frac{dY}{dt} = (\text{rate in}) - (\text{rate out})}$$

E14 The RL - Circuit

Applying Kirchhoff's Rules & a defⁿ or two we find



$$\mathcal{E} = V_R + V_L \quad \text{(Kirchhoff)}$$

$$V_L = L\frac{dI}{dt} \quad \text{(Resists change in curr}$$

$$V_R = IR \quad \text{(Ohm's Law)}$$

$$\mathcal{E} = IR + L\frac{dI}{dt}$$

Now if $\mathcal{E}$ = constant and $I(0) = 0$ find $I(t)$,

$$\frac{\mathcal{E} - IR}{L} = \frac{dI}{dt} \quad \therefore \quad \int \frac{dI}{\mathcal{E} - IR} = \int \frac{dt}{L}$$

$$\int \frac{dI}{\mathcal{E} - IR} = \frac{-1}{R}\ln(\mathcal{E} - IR)$$

$$\int \frac{dt}{L} = \frac{t}{L} + C$$

Thus $\frac{-1}{R}\ln(\mathcal{E} - IR) = \frac{t}{L} + \tilde{C} \Rightarrow \mathcal{E} - IR = \tilde{C}e^{-\frac{R}{L}t}$ that is

$$I = \frac{\mathcal{E}}{R}\left(1 + ce^{-\frac{R}{L}t}\right)$$

$$I(0) = \frac{\mathcal{E}}{R}(1 + c) = 0 \quad \therefore \quad c = -1$$

$$\therefore \boxed{I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

Remark: the limiting current as $t \to \infty$ is $\mathcal{E}/R$. That is physically speaking the inductor is a short circuit for "long" times ($\tau = L/R$ then $5\tau \approx \infty$ pragmatically speaking.)

Exponential Growth & Decay

If the growth of a population P is proportional to its size then

$$\frac{dP}{dt} = kP$$

Likewise if the rate of change of Y is proportional to Y

$$\frac{dy}{dt} = kY$$

As discussed in $\boxed{E1}$ of §7.3

$$\boxed{Y(t) = Y_o e^{kt} \quad \text{where} \quad Y_o = Y(0).}$$

Likewise $P(t) = P_o e^{kt}$ where $P_o = P(0)$. Both follow simply from sep. of variables. By the way:

$$\frac{dP}{dt} = kP \iff \frac{1}{P}\frac{dP}{dt} = k \iff \text{relative growth rate constant}$$

So $k \equiv$ the relative growth rate.

$\boxed{E15}$ If the population doubles every 10 yrs what is k?

$$P(0) = P_o \quad \text{and} \quad P(10) = 2P_o = P_o e^{10k}$$

$$\Rightarrow \ln(2) = 10k \Rightarrow k = \frac{\ln(2)}{10\text{yr.}} = 0.0693\frac{1}{\text{yr.}}$$

The relative growth rate is 6.93%.

$\boxed{E16}$ Let $Y(t) = m(t)$ be the mass of some radioactive substance then as the mass destabalizes via radiation we have

$$\frac{dm}{dt} = km \qquad (k < 0 \text{ since } m \text{ is decreasing})$$

$$\Rightarrow m(t) = m_o e^{kt}$$

If balonium has a half-life of 1yr. then what percentage of the balonium is still in the fridge after $\frac{1}{10}$ yr.?

$$m(1) = \frac{m_o}{2} = m_o e^{k} \therefore k = \ln(1/2) = \underline{-0.693 = k}$$

$$\therefore m\left(\tfrac{1}{10}\right) = m_o e^{-0.693\left(\frac{1}{10}\right)} = (0.933)m_o \Longrightarrow \boxed{93.3\% \text{ remains}}$$

THE LOGISTIC EQ$^{\underline{n}}$

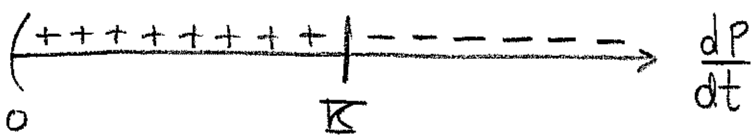
E17 This is another model for population growth, the basic idea is that when the population P is small then $\frac{dP}{dt} = kP$ but as P gets big the resources are all used up and the population is unable to continue growing past some limiting population $K \equiv$ the carrying capacity. The simplest eq$^{\underline{n}}$ encoorporating the above features is

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \quad \text{The Logistic Eq}^{\underline{n}}}$$

Notice that as $P \longrightarrow K$ we have $\frac{dP}{dt} \longrightarrow 0$. As we desired the growth slows to zero as we approach the carrying capacity. Additionally when $P \ll K$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \cong kP$$

So for small population this model is like exponential growth. Now lets figure out what general features the sol$^{\underline{n}}$'s to the Logistic Eq$^{\underline{n}}$ must have, (time for some calc. I)



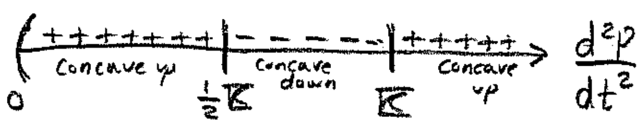
P increases when $P < K$
P decreases when $P > K$

What about concavity? Lets differentiate,

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt}\left(1 - \frac{P}{K}\right) - \frac{k}{K}P\frac{dP}{dt}$$

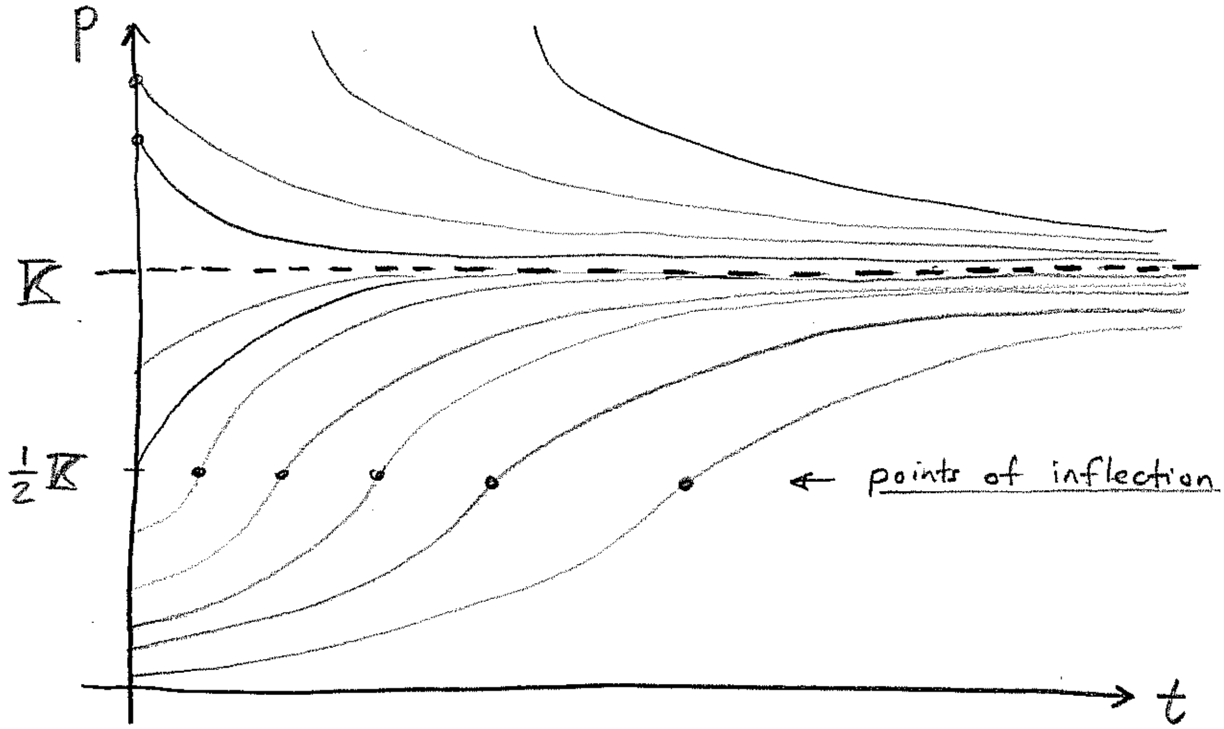
$$= k\left(1 - \frac{2P}{K}\right)\frac{dP}{dt}$$

$$= k^2\left(1 - \frac{2P}{K}\right)\left(1 - \frac{P}{K}\right)$$



Notice $\frac{dP}{dt}$ is maximized at $P = \frac{1}{2}K$.

Graph of Sol^{n}'s to Logistic Eq^{n}



Inevitably as $t \to \infty$ the sol^{n} goes to K no matter what the intial condition was.

Remark: We have yet to find a sol^{n}. Next we'll explicitly solve the Log. Eq^{n}. I think its interesting we can see so much just from studying the DEq^{n} directly.

E17 conclusion **Analytical, nongraphical solution**

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \Rightarrow \frac{dP}{P(1-P/K)} = k\,dt$$

Now to integrate in P we'll use partial fractions,

$$\frac{1}{P(1-P/K)} = \frac{A}{P} + \frac{B}{1-P/K}$$

$$1 = A(1-P/K) + BP \quad \xrightarrow{P=0} A = 1, \quad \xrightarrow{P=K} 1 = BK \therefore B = 1/K$$

Thus $\frac{1}{P(1-P/K)} = \frac{1}{P} + \frac{1}{K-P}$

$$\int \frac{dP}{P(1-P/K)} = \int\left(\frac{1}{P} + \frac{1}{K-P}\right)dP = \ln|P| - \ln|K-P| = \ln\left|\frac{P}{K-P}\right|$$

$$\int k\,dt = kt + C$$

Hence $\ln\left|\frac{P}{K-P}\right| = kt + C \Rightarrow \left|\frac{P}{K-P}\right| = e^c e^{kt} \Rightarrow \frac{P}{K-P} = Ae^{kt}$

$A = \pm e^c$

Now solve for P

$$P = (K-P)Ae^{kt}$$

$$P(1+Ae^{kt}) = AKe^{kt} \Rightarrow P = \frac{AKe^{kt}}{1+Ae^{kt}} = \boxed{\frac{K}{1+Ae^{-kt}} = P(t)}$$

> **Exercise**: Verify for yourself that the conclusions we reached for inc/dec concave up/down exc... are duplicated by this sol$^{\underline{n}}$.

Remark: What ever the initial population is the final population is K

$$\lim_{t\to\infty}\left(\frac{K}{1+A\underbrace{e^{-kt}}_{\to 0}}\right) = K$$

Remark: the graphical analysis carried out for this differential eq$^{\underline{n}}$ can be applied to many others. However, our focus is primarily algebraic.

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E18 Suppose that $\frac{dP}{dt} = 0.05P - 0.0005P^2$

Then what is the carrying capacity K ? and k ?

$$\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{100}\right) = kP\left(1 - \frac{P}{K}\right)$$

Comparing we identify $\boxed{K = 100 \text{ and } k = 0.05}$

E19 Suppose the carrying capacity of the US is 1000 (million). Additionally in 1990 $P = 250$ and in 2000 $P = 275$ million. Find $P(t)$ then predict the pop. in 2010 and 2100.

$$P(t) = \frac{1000}{1 + Ae^{-kt}}$$

Let 1990 be $t = 0$, then $P(0) = \frac{1000}{1+A} = 250 \Rightarrow \boxed{A = 3}$

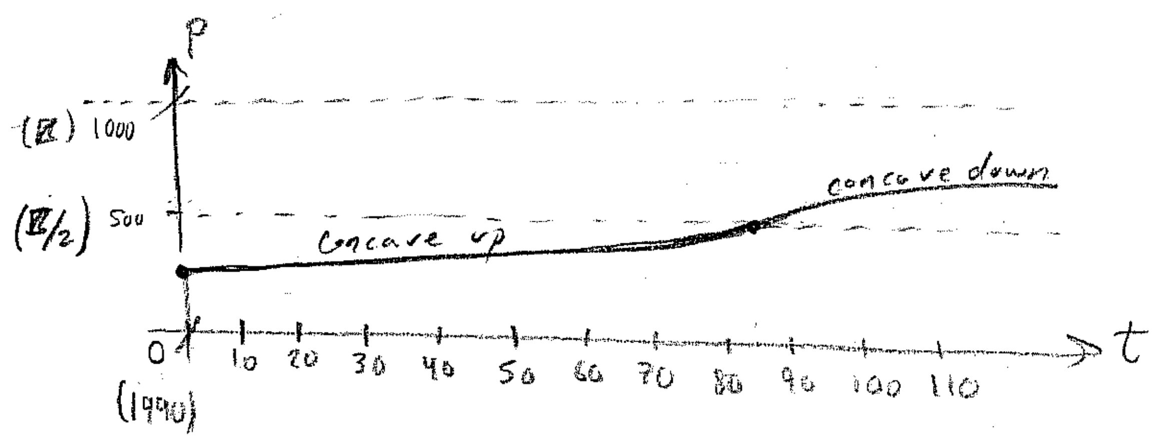
Additionally: $P(10) = \frac{1000}{1+3e^{-10k}} = 275 \Rightarrow 725 = 275(3e^{-10k})$

$$\Rightarrow \frac{725}{3 \cdot 275} = e^{-10k} = \frac{29}{33}$$

$$\Rightarrow \boxed{k = \frac{\ln(29/33)}{-10} = 0.01292}$$

$$P(20) = \frac{1000}{1+e^{-0.13(20)}} = 301 \text{ million in } 2010$$

$$P(110) = \frac{1000}{1+e^{-0.13(110)}} = 580 \text{ million in } 2100$$



$$P(t) = \frac{K}{2} = \frac{K}{1+3e^{-kt}} \Rightarrow 2 = 1 + 3e^{-kt}$$

$$\frac{1}{3} = e^{-kt} \Rightarrow t = \frac{\ln(3)}{k} = \frac{\ln(3)}{0.01292} = 85 \Rightarrow P(85) = K/2$$