# THE DIRAC DELTA "FUNCTION"



**Def$^{n}$/** The DIRAC delta function $\delta(t)$ is characterized by

1.) $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$

2.) $\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$

for any function $f$ that is continous on some nbhd of zero.

Technically $\delta(t)$ is a "generalized function" or better yet a "distribution". It was introduced by P. A. M. DIRAC for physics, but was only later justified by mathematicians. This trend is also seen in recent physics, the physics community tends to do calculations that are not well-defined. Fortunately, physical intuition has guided them to not make very bad mistakes for the most part. DIRAC later introduced something that came to be named "DIRAC STRING" to describe a quirk in the mathematics of the magnetic monopole. It took 20+ years for the mathematics to really catch-up and better, explain the DIRAC STRING in terms of a beautiful mathematical construction called fiber bundles. I digress! Any way, we sometimes say "we have it down to a science", it'd be better to say "we have it down to a math".

**Remark:** DIRAC DELTA FUNCTIONS in 3-dimensions work about the same $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$ for $\vec{r} = (x, y, z)$. One application is to model the ideal point charge $q$ at location $\vec{a}$ it has charge density

$$\rho(\vec{r}) = q\,\delta(\vec{r} - \vec{a})$$

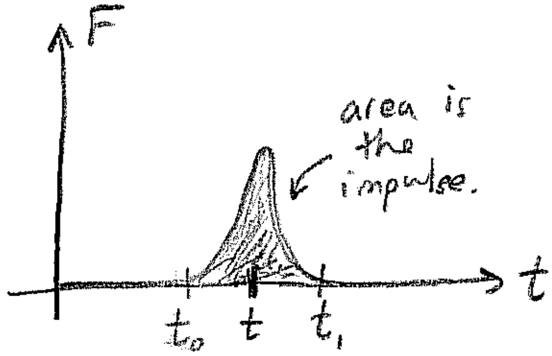
See Griffith's Electrodynamics or take PY 415 for lots of this stuff.

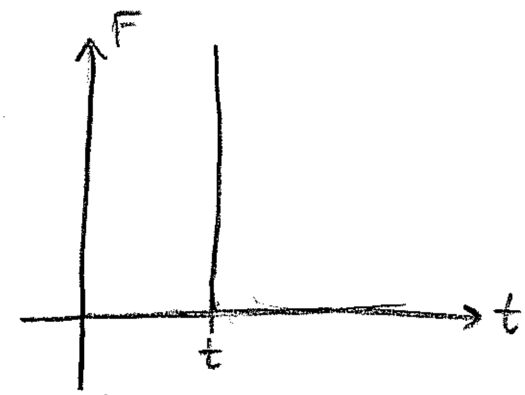
## Heuristic Justification of $\delta(t)$

Impulse is defined to be the time integral of force experienced by some object over some short time interval $t_0 \to t_1$

$$\text{Impulse} \equiv \int_{t_0}^{t_1} F(t)\,dt = \int_{t_0}^{t_1} \frac{dP}{dt}\,dt = P(t_1) - P(t_0)$$

Since $F = \frac{dP}{dt}$ it is also equal to the change in momentum. You might think of a hammer striking a nail or a ball bouncing off a wall. A large force is exerted over a short time period.



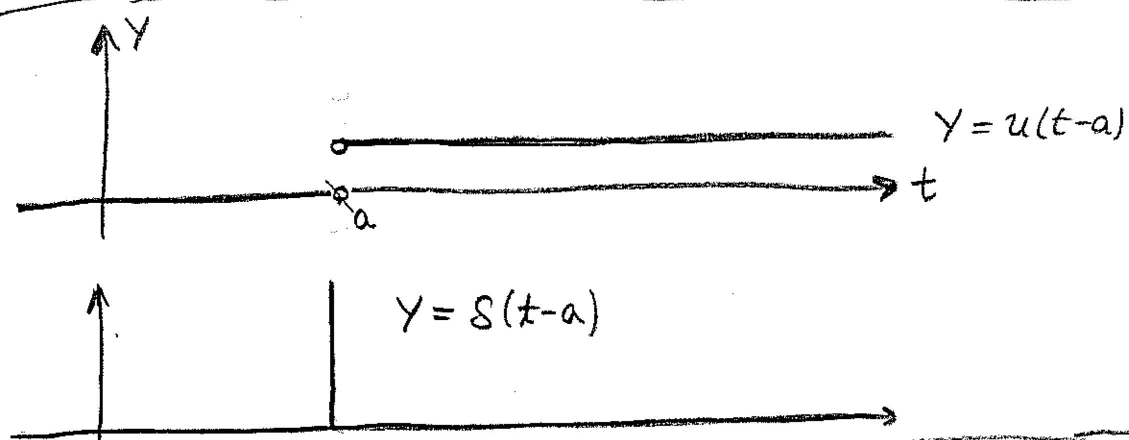

You can imagine applying a greater force over a shorter time till in the limit you approach the notion of the delta function. The DIRAC DELTA function can be used to model an impulse where we do not perhaps know the details of the impact, but we know it happened quickly and with a certain overall change in momentum. In such case the $\delta(t)$ provides a useful idealization. Similarly when it is used to describe charge densities, it provides us a convenient mathematics for describing a localized source. I think a point charge is not really a point, but rather an extended (although tiny) body. We don't know the details of such tiny things, or if we do they're complicated. In such cases the DIRAC delta function provides a useful mathematical idealization.

$$\boxed{\text{Proposition (6): } \mathcal{L}\{\delta(t-a)\}(s) = e^{-as}}$$

**Proof:** integrals with $\delta$-fncts are easy, integration becomes evaluation,

$$\mathcal{L}\{\delta(t-a)\}(s) = \int_0^\infty e^{-st}\,\delta(t-a)\,dt$$
$$= e^{-st}\Big|_{t-a=0}$$
$$= e^{-as}.$$

Consider the graph of the unit step function $u(t-a)$

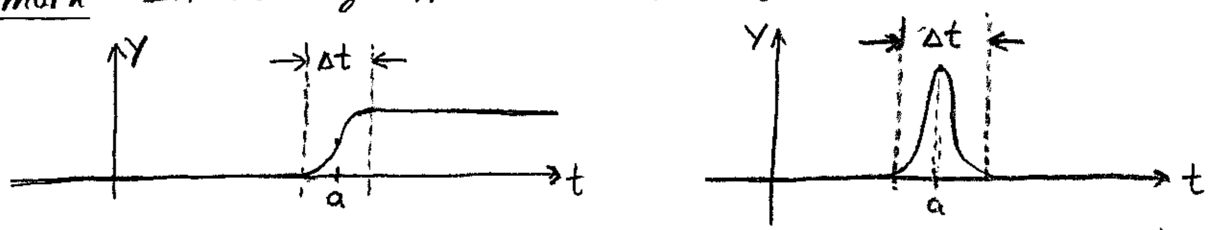



The unit step is constant everywhere except at $t=a$ where it jumps 1 in zero time. This suggests,

$$\frac{d}{dt}\left[u(t-a)\right] = \delta(t-a)$$

Of course there is something exotic here, $u(t-a)$ is not even continuous, what right have we to differentiate it? Of course the result $\delta(t-a)$ is not a function and the $\frac{d}{dt}$ here is a more general idea that the one seen in Calc I.

**Remark:** In reality applications typically have functions more like




As $\Delta t \to 0$ we obtain the unit step function and $\delta(t)$. We use $u(t-a)$ and $\delta(t-a)$ because we either don't know or don't care the details of what happens in $\Delta t$.

**E144** Consider mass spring system hit by hammer at $t=0$ this can be described by (no friction & $m=k=1$)

$$x'' + x = \delta(t)$$

Suppose the system has $x(0)=0$ & $x'(0)=0$. Then

$$s^2 X + X = \mathcal{L}\{\delta(t)\}(s) = e^0 = 1$$

$$(s^2+1)X = 1$$

$$X = \frac{1}{s^2+1} \Rightarrow \boxed{x(t) = \sin t}$$

- Notice that while $x(0)=0$, $x'(0) = 1 \neq 0$. This is to be expected as $x'(0^-)=0$ while $x'(0^+)=1$ since $\Delta P = m\Delta V = 1$ so the velocity has to change very quickly.

**E145** $x'' + 9x = 3\delta(t-\pi)$, $x(0)=1$ and $x'(0)=0$

$$s^2 X - s + 9X = 3e^{-\pi s}$$

$$(s^2+9)X = 3e^{-\pi s} + s \Rightarrow X = 3e^{-\pi s}\frac{1}{s^2+9} + \frac{s}{s^2+9}$$

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\{X\}(t) \\
&= \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{3}{s^2+9}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}(t) \\
&= \sin(3(t-\pi))\,u(t-\pi) + \cos(3t) \\
&= (\sin 3t \underbrace{\cos 3\pi}_{-1} - \underbrace{\sin 3\pi}_{0} \cos 3t)\,u(t-\pi) + \cos(3t) \\
&= \boxed{\begin{cases} \cos 3t & t < \pi \\ \cos 3t - \sin 3t & \pi < t \end{cases}}
\end{aligned}$$

E146 Solve the initial value problem given below:

$$y'' - 2y' + y = u(t-1) \qquad y(0)=0 \;,\; y'(0)=1$$

$$s^2 Y - 1 - 2sY + Y = e^{-s}/s$$

$$(s^2 - 2s + 1)\,Y = 1 + \frac{1}{s}e^{-s}$$

$$Y = \underbrace{\frac{1}{s^2-2s+1}}_{\text{II}} + \boxed{\underbrace{\frac{1}{s(s^2-2s+1)}}_{F(s)} e^{-s}}_{\text{I}} \longleftarrow \text{think. Th}^{\underline{m}}\ (8)$$

I

$$\frac{1}{s(\underbrace{s^2-2s+1}_{(s-1)^2})} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$1 = A(s-1)^2 + B(s-1)s + Cs$$

$s=1|$ $\quad 1 = C$

$s=0|$ $\quad 1 = A$

$s=2|$ $\quad 1 = A + 2B + 2C = 1 + 2B + 2$

$$2B = -2 \quad \therefore \quad B = -1$$

$$F(s) = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}$$

$$f(t) = 1 - e^t + te^t$$

$$\mathcal{L}^{-1}\{F(s)e^{-s}\} = f(t-1)\,u(t-1) = \left(1 - e^{t-1} + (t-1)e^{t-1}\right)u(t-1)$$

II

$$\frac{1}{s^2-2s+1} = \frac{1}{(s-1)^2} \longrightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+1}\right\} = te^t.$$

$$\boxed{Y(t) = te^t + \left(1 - e^{t-1}(t-2)\right)u(t-1)}$$

E147 Solve the initial value problem:

$$y'' + 5y' + 6y = u(t-1) \quad \text{with } y(0)=2, \quad y'(0)=1$$

$$s^2 Y - 2s - 1 + 5(sY - 2) + 6Y = \frac{1}{s}e^{-s}$$

$$(s^2+5s+6)Y = 2s+1+10 + \frac{1}{s}e^{-s}$$

$$Y(s) = \underbrace{\frac{2s+11}{s^2+5s+6}}_{\text{I}} + \underbrace{\frac{1}{s(s^2+5s+6)}}_{F(s)} e^{-s}$$

I $$\frac{2s+11}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$2s+11 = A(s+3) + B(s+2)$$

$s=-3)$ $5 = -B$ $\therefore$ $B = -5$

$s=-2)$ $7 = A$

E147 Continued

$$F(s) = \frac{1}{s(s^2+5s+6)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$1 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$$

$s=-2)$ $1 = -2B$ $B = -1/2$

$s=-3)$ $1 = 3C$ $C = 1/3$

$s=0)$ $1 = 6A$ $A = 1/6$

$$f(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}$$

using the values for A, B, C and take $\mathcal{L}^{-1}\{F\}$.

Then, using ① and preparing to use $f(t)$ for the $F(s)$ factor,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{7}{s+2} - \frac{5}{s+3}\right\}(t) + \mathcal{L}^{-1}\left\{F(s)\,e^{-s}\right\}(t)$$

$$= 7e^{-2t} - 5e^{-3t} + f(t-1)\,u(t-1)$$

$$= \boxed{7e^{-2t} - 5e^{-3t} + \left(\frac{1}{6} - \frac{1}{2}e^{-2(t-1)} + \frac{1}{3}e^{-3(t-1)}\right)u(t-1)}$$

E148 Solve the initial valued problem:

$$y'' - 7y' + 12y = 12u(t-4) \quad , \quad y(0) = 1$$
$$y'(0) = 4$$

this makes it nicer.

$$s^2 Y - s - 4 - 7(sY - 1) + 12Y = \frac{12}{s}e^{-4s}$$

$$(s^2 - 7s + 12)Y = s - 3 + 12\left(\frac{e^{-4s}}{s}\right)$$

$$Y = \frac{s-3}{s^2-7s+12} + \frac{12}{s(s^2-7s+12)}e^{-4s}$$

$$Y = \frac{1}{s-4} + F(s)e^{-4s}$$

need to work on this.

$\boxed{E148}$ Continued:

$$F(s) = \frac{12}{s(s-3)(s-4)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-4}$$

$$1 = A(s-3)(s-4) + B(s-4)s + Cs(s-3)$$

| | | |
|---|---|---|
| $s=0$ | $12 = 12A$ | $A = 1$ |
| $s=3$ | $12 = -3B$ | $B = -4$ |
| $s=4$ | $12 = 4C$ | $C = 3$ |

$$F(s) = \frac{1}{s} - \frac{4}{s-3} + \frac{3}{s-4}$$

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = 1 - 4e^{3t} + 3e^{4t}.$$

Finally then,

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + \mathcal{L}^{-1}\{F(s)e^{-4s}\}$$
$$= e^{4t} + f(t-4)\,u(t-4)$$
$$= e^{4t} + \left(1 - 4\exp(3(t-4)) + 3\exp(4(t-4))\right)u(t-4$$

$$\therefore \boxed{y(t) = e^{4t} + \left(1 - 4e^{3(t-4)} + 3e^{4(t-4)}\right)u(t-4)}$$

E149 (E148, With the messy initial conditions)

$$y'' - 7y' + 12y = 12u(t-4) \qquad y(0)=1, \quad y'(0) = \tfrac{1}{2}$$

$$s^2 Y - s - \tfrac{1}{2} - 7(sY - 1) + 12Y = \frac{12}{s}e^{-4s}$$

$$(s^2 - 7s + 12)Y = s + \tfrac{1}{2} - 7 + \frac{12}{s}e^{-4s} = s - \frac{13}{2} + \frac{12}{s}e^{-4s}$$

$$Y = \underbrace{\frac{s - 13/2}{s^2 - 7s + 12}}_{\mathrm{I}} + \underbrace{\frac{12}{s(s^2 - 7s + 12)}e^{-4s}}_{\mathrm{II}}$$

In view of the factoring $s^2 - 7s + 12 = (s-3)(s-4)$

$$\mathrm{I} = \frac{s - 13/2}{(s-3)(s-4)} = \frac{A}{s-3} + \frac{B}{s-4}$$

$$s - 13/2 = A(s-4) + B(s-3)$$

$\underline{s=4|}\quad 4 - 13/2 = -5/2 = B$

$\underline{s=3|}\quad 3 - 13/2 = -7/2 = -A$

$$\frac{s - 13/2}{(s-3)(s-4)} = \frac{7/2}{s-3} - \frac{5/2}{s-4}$$

$$\mathrm{II} = \underbrace{\frac{12}{s(s-3)(s-4)}}_{F(s)} e^{-4s}$$

$$F(s) = \frac{12}{s(s-3)(s-4)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-4}$$

$$12 = A(s-3)(s-4) + Bs(s-4) + Cs(s-3)$$

$\underline{s=0)}\quad 12 = 12A \quad \therefore\ A = 1$

$\underline{s=3|}\quad 12 = -3B \quad \therefore\ B = -4$

$\underline{s=4|}\quad 12 = 4C \quad \therefore\ C = 3$

Thus $F(s) = \frac{1}{s} - \frac{4}{s-3} + \frac{3}{s-4} \Rightarrow f(t) = 1 - 4e^{3t} + 3e^{4t}$

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\mathrm{I}\} + \mathcal{L}^{-1}\{\mathrm{II}\} = \frac{7}{2}e^{3t} - \frac{5}{2}e^{4t} + \mathcal{L}^{-1}\{F(s)e^{-4s}\}$$

$$\Rightarrow \boxed{y(t) = \frac{7}{2}e^{3t} - \frac{5}{2}e^{4t} + \left(1 - 4e^{3(t-4)} + 3e^{4(t-4)}\right)u(t-4)}$$