Series Solutions to Linear ODEs : Chapter 8

In calculus II we learned that many functions can be locally represented by a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ on some Interval of Convergence (IOC).

Defn/ If f is a function and $I \subset dom(f)$ then f is <u>analytic</u> on I if $f(x) = \sum_{n=0}^{\infty} c_n (x-x_o)^n$ for some $x_o \in I$. In other words, f is analytic on I if it has a power series representation over all of I

E150 Let $f(x) = \frac{1}{1-x}$ then recall the geometric series result said $a + ar + ar^2 + \dots = \frac{a}{1-r}$ provided $|r| < 1$. Thus,

$$\boxed{f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots} \quad \left(\begin{array}{l}\text{identified} \\ a=1,\ r=x\end{array}\right)$$

Notice the power series representation given above remains valid for all $x \in (-1,1)$. <u>Therefore, f analytic on $(-1,1)$.</u>

<u>Remark</u>: you should review calculus II and refresh your memory on the many uses and tricks involving the geometric series. (I have posted a few pdfs of my notes from math 132)

<u>Proposition</u>: If f is analytic on I with power series representation $f(x) = \sum_{n=0}^{\infty} c_n (x-x_o)^n$ for some $x_o \in I$ then f is smooth on I and the derivatives of f at x_o are given by $n!\, c_n = f^{(n)}(x_o)$. In other words if f is analytic on I then it converges to it's Taylor Series,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!} (x-x_o)^n$$

E151 Known Maclaurin series from Calculus II,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

It can be shown $\sin(x)$, $\cos(x)$ and e^x are analytic over $\mathbb{R}$. This means the formulas above hold for all $x \in \mathbb{R}$.

QUESTION: What do I mean when I say a "function converges to it's Taylor Series representation on I"? Notice if $f(x)$ has derivatives at $x_0 \in I$ for all orders then we can construct a Taylor Series for the function

$$T^f(x) = \underbrace{\underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{best linear approx. of } f(x) \text{ at } x=x_0} + \frac{1}{2} f''(x_0)(x-x_0)^2}_{\text{best quadratic approximation of } f(x) \text{ centered at } x=x_0} + \cdots$$

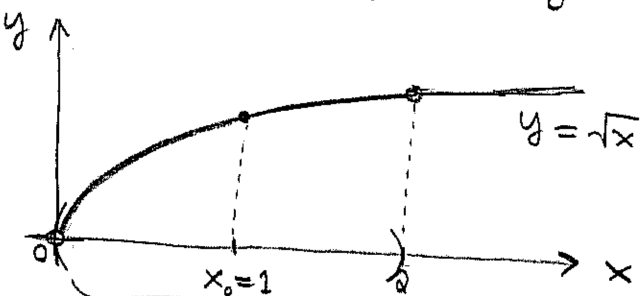
When we say $T^f \longrightarrow f$ on I we mean $T^f(x) = f(x)$ for all $x \in I$. Define $T_n^f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$. Here $T_n^f(x)$ is the n-th Taylor Polynomial. Another way to express analyticity on I is $f(x) = \lim_{n\to\infty} T_n^f(x)$ for all $x \in I$. For most functions a direct proof of analyticity is rather involved however, Lagrange provided a nice guide for bounding the error $\mathcal{E}_n(x) = f(x) - T_n^f(x)$ in the n^{th} Taylor polynomial,

$$\mathcal{E}_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \quad \text{for some } \xi \in (x_0, x).$$

E152 A function which is not analytic at zero is $f(x)=\sqrt{x}$. Notice $f'(x) = \frac{1}{2\sqrt{x}}$ is not defined at $x_o = 0$ hence f is not analytic at $x_o = 0$. However, $f(x) = \sqrt{x}$ is analytic on $(0, \infty)$. For example, at $x_o = 1$ we have the following power series representation for $\sqrt{x}$, $f''(x) = \frac{-1}{4x^{3/2}}$

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \cdots$$

the I.O.C. includes $(0, 2)$, it may or may not include $x = 0, 2$.



this is the largest possible I.O.C. since larger intervals symmetric about $x_o = 1$ will go past zero.

> **Claim:** If we find power series expansion about $x_o = 4$ then $(0, 8) \subset$ I.O.C.

Th^m/ The I.O.C. for a power series will be one of the following cases given $f(x) = \sum_{n=0}^{\infty} a_n (x-x_o)^n$,

(I.) I.O.C. $= \mathbb{R}$

(II.) I.O.C. $= (x_o - R, x_o + R)$ or $(x_o - R, x_o + R]$ or $[x_o - R, x_o + R)$ or $[x_o - R, x_o + R]$, I call $x_o \pm R$ the <u>endpoints</u> of the I.O.C.

(III.) I.O.C. $= \{x_o\}$

<u>Remark</u>: you should recall the Ratio Test <u>usually</u> helps use determine the open I.O.C. then we need other tools to check the endpoints.

<u>Remark</u>: for a given function $f(x)$ we usually have many points where f is analytic. Each point gets a different series expansion and the I.O.C. will likewise change from pt. to pt.

Power Series Calculations:

The last couple pages were just about <u>where</u> and <u>how</u> a power series can represent a function. We now turn to the question of how to equate, add, subtract multiply and divide power series. In short, just like polynomials, except power series keep going and going and going ....

<u>Proposition</u>: $\sum_{n=0}^{\infty} a_n (x-x_o)^n = 0$ for all $x \in (a,b)$ iff $a_n = 0$ for all n.

<u>Proposition</u>: $\sum_{n=0}^{\infty} a_n (x-x_o)^n = \sum_{n=0}^{\infty} b_n (x-x_o)^n \ \forall x \in (a,b)$ iff $a_n = b_n \ \forall n$.

<u>Remark</u>: the propositions above simply say we can equate coefficients of like powers in an eq$^{\underline{n}}$ with power series. However, it is important we have all terms centered about same x_o, also we need to match indices to make manifest like powers of x.

E153 Find relation between coefficients a_n and b_k given that
$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=1}^{\infty} b_k x^{k+3}$$

I find it helps to write out several terms to begin,
$$a_o + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = b_1 x^4 + b_2 x^5 + \cdots .$$
It follows $a_o = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = b_1$ etc...
Since $n = 0, 1, 2, 3$ are all trivial for a_n we can write,
$$\sum_{n=4}^{\infty} a_n x^n = \sum_{k=1}^{\infty} b_k x^{k+3}$$

index shifting: let $m = k+3$ then $k = m-3$ and when $k = 1 \Rightarrow m = 4$.

$$= \sum_{m=4}^{\infty} b_{m-3} x^m$$

Now switch m to n to find that $\sum_{n=4}^{\infty} a_n x^n = \sum_{n=4}^{\infty} b_{n-3} x^n$ hence $a_n = b_{n-3}$ for $n \geq 4$ and $a_n = 0$ for $n \leq 3$.

INDEX SHIFTING CALCULATIONS : §8.2

- Write out first few terms, drop exceptional terms from Σ if need be.
- Look for common power of x between terms.
- Change index so that all power series have same index structure on x.

E154 Express $\sum_{n=0}^{\infty} 2^n x^{n+1} + \sum_{k=2}^{\infty} (k+6)x^{k+2}$ as a single summation and possibly a few exceptional terms

To begin write out the 1st few terms to see how the low-order terms match-up,

$$\sum_{n=0}^{\infty} 2^n x^{n+1} = \underbrace{x + 2x^2 + 4x^3}_{\text{exceptional terms.}} + 8x^4 + \cdots$$

$$\sum_{k=2}^{\infty} (k+6)x^{k+2} = 8x^4 + 9x^5 + 10x^6 + \cdots .$$

I'll rewrite the first term to match the 2nd,

$$\sum_{n=0}^{\infty} 2^n x^{n+1} = x + 2x^2 + 4x^3 + \underbrace{\sum_{n=3}^{\infty} 2^n x^{n+1}} = x + 2x^2 + 4x^3 + \sum_{k=2}^{\infty} 2^{k+1} x^{k+2}$$

want to match this with the sum with x^{k+2}. This means we want $n+1 = k+2$ thus $n = k+1$. Also when $n=3 \Rightarrow k=2$.

Return to the task of adding the power series,

$$\sum_{n=0}^{\infty} 2^n x^{n+1} + \sum_{k=2}^{\infty} (k+6)x^{k+2} = x + 2x^2 + 4x^3 + \sum_{k=2}^{\infty} 2^{k+1} x^{k+2} + \sum_{k=2}^{\infty} (k+6)x^{k+2}$$

$$= x + 2x^2 + 4x^3 + \sum_{k=2}^{\infty} \left(2^{k+1} + k + 6\right)x^{k+2}.$$

Remark: these sort of calculations occur naturally as we use power series to solve ODEs. I'm just trying to throw in a few examples to warm-up to those more meaningful calculations. See §8.2 for more nice examples.

E155 Find 1st 4 nontrivial terms in product below,

$$\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}\right) = \left(1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\dots\right)\left(1+2x+2x^2+\tfrac{4}{3}x^3+\dots\right)$$

$$\begin{aligned} = 1 + 2x + 2x^2 &+ \tfrac{4}{3}x^3 + \dots \\ + x + 2x^2 &+ 2x^3 + \dots \\ + \tfrac{1}{2}x^2 &+ x^3 + \dots \\ &+ \tfrac{1}{6}x^3 + \dots \end{aligned}$$

$$= \underline{1 + 3x + \tfrac{9}{2}x^2 + \tfrac{27}{6}x^3 + \dots}.$$

note I can ignore x^4 and higher because I'm only asked for 1st 4 nontrivial terms

For this particular example I can even tell you the general formula for an arbitrary coefficient in the product, we can show,

$$\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}\right) = \sum_{k=0}^{\infty}\frac{3^k}{k!}x^k.$$

Remark: text formalizes this via the "Cauchy Product"

Usually it's not so easy to see such a formula. Do you see how I knew the product would be $\sum_{k=0}^{\infty}\frac{(3x)^k}{k!}$?

E156 Let $f(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ find power series representation of $\int f(x)\,dx$ centered about $x_o = 1$.

$$\int f(x)\,dx = \int\left(\sum_{n=0}^{\infty} a_n (x-1)^n\right)dx$$

$$= \underline{C + \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-1)^{n+1}}.$$

we can integrate power series term by term. The open I.O.C. is unaltered but the endpts can be gained or lost as a result of integrating.

E157 Find power series sol^{n} of $\int x^3 \sin(x^2)\,dx$.

$$\int x^3\sin(x^2)\,dx = \int x^3 \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x^2)^{2n+1}\,dx$$

$$= \int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{4n+5}\right)dx$$

$$= \underline{C + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\,(4n+6)}x^{4n+6}}.$$