HOW TO SOLVE LINEAR DEq$^{\underline{n}}$S VIA POWER SERIES:

Let me give the general idea: given a DEq$^{\underline{n}}$ we put all parts of the DEq$^{\underline{n}}$ into a power series representation (usually about $x_o = 0$). Then we <u>assume</u> our sol$^{\underline{n}}$ has the form $y = \sum_{n=0}^{\infty} a_n x^n$. The coefficients a_n are undetermined at the beginning, but once we substitute the series into the DEq$^{\underline{n}}$ we find conditions on a_n. If the DEq$^{\underline{n}}$ is 1st order we'll be able to write all coefficients in terms of a single arbitrary coefficient (usually a_o). If it's 2nd order we'll end up with a_o, a_1 determining all other coefficients.

$\boxed{\text{E158}}$ Solve $\dfrac{dy}{dx} = x^3 \sin(x^2)$.

Assume $y = \sum_{k=0}^{\infty} a_k x^k$ then $\dfrac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1}$ ($k=0$ trivial)

Now substitute and put in power series rep. of $x^3 \sin(x^2)$,

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+5}$$

Write out a few terms,

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots =$$

$$= x^5 - \frac{1}{3} x^9 + \cdots$$

Clearly $a_1 = a_2 = a_3 = a_4 = a_5 = 0$ & $a_6 = 1/6$, $a_7 = a_8 = a_9 = 0$, $a_{10} = \frac{-1}{30}$ etc.

Actually, this problems kinda tricky, the 4 in $4n+5$ makes it hard to match up the terms. However, it is clear enough that

$$a_6, a_{10}, a_{14}, \ldots, a_{4n+6} \neq 0 \text{ for } n \geq 0 \text{ otherwise } a_k = 0 \text{ for } k > 0.$$

$$a_6 = \frac{1}{6}, \quad a_{10} = \frac{-1}{10 \cdot 3!}, \quad a_{14} = \frac{1}{14 \cdot 5!}, \ldots, \quad a_{4n+6} = \frac{(-1)^n}{(2n+1)!(4n+6)}$$

Thus $\boxed{y = a_o + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(4n+6)} x^{4n+6}}$, note a_o plays role of "C" in $\boxed{\text{E157}}$

Remark: $\boxed{E158}$ is a pretty awful example to start with. I probably skipped to $\boxed{E159}$ in lecture because it illustrates the method for <u>homogeneous</u> DEq^{n}'s, technically $\frac{dy}{dx} = x^3 \sin(x^2)$ is <u>nonhomogeneous</u>, also obviously $\boxed{E157}$ gives a better sol^{n} to the problem.

$\boxed{E159}$ Solve $\frac{dy}{dx} = 2y$ with power series technique.

Assume $y = \sum_{n=0}^{\infty} a_n x^n$ then $\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}$ substitute into $\frac{dy}{dx} = 2y$ to find, (switched n to k to make shift ② easier)

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 2\sum_{k=0}^{\infty} a_k x^k \overset{②}{=} \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1}$$

$k = n-1$
$n = k+1$
$k = 0 \to n = 1$

$\Rightarrow$ <u>for $n \geq 1$ we find $n a_n = 2a_{n-1}$</u>,

This is a <u>recurrence</u> <u>relation</u> for a_n, it gives an <u>iterative</u> formula to determine a_n, we may or may not be able to solve a recurrence relation to yield an explicit general formula for a_n. It turns out we find a nice formula for a_n here,

$$a_n = \frac{2}{n}a_{n-1} \xrightarrow{n=1} a_1 = 2a_0$$
$$\xrightarrow{n=2} a_2 = \frac{2}{2}a_1 = \frac{2^2}{2}a_0$$
$$\xrightarrow{n=3} a_3 = \frac{2}{3}a_2 = \frac{2^3}{3\cdot 2}a_0$$
$$\xrightarrow{n=4} a_4 = \frac{2}{4}a_3 = \frac{2^4}{4!}a_0$$

We observe a <u>pattern</u> here; <u>$a_n = \frac{2^n}{n!}a_0$ for $n \geq 0$</u>.

Thus,
$$\boxed{y = \sum_{n=0}^{\infty} \frac{a_0 2^n}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = a_0 e^{2x}}$$

<u>Remark</u>: this is not surprising, $y' = 2y \Rightarrow y' - 2y = 0$ yielding Char. Eq^{n} $\lambda - 2 = 0 \therefore \lambda = 2 \Rightarrow$ <u>$y = c_1 e^{2x}$</u>. The a_0 is filling the same role as c_1.

<u>Remark:</u> $\boxed{E159}$ intends to illustrate the method, obviously it is not the best way to solve $\frac{dy}{dx} = 2y$. The real beauty of Chapter 8 is we can solve DEq^{n}'s which are not solved by previous methods.

$\boxed{E160}$ Solve $y'' + y = 0$ via power series. Again this is to illustrate the method on a nice (easy) example

Let $y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Thus,

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{\substack{k = n-2 \\ n=2 \to k=0 \\ n = k+2}} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\text{let } n=k} = 0$$

$$\sum_{k=0}^{\infty} \left((k+2)(k+1) c_{k+2} + c_k \right) x^k = 0$$

We find the recurrence relation $c_{k+2} = \dfrac{-1}{(k+2)(k+1)} c_k$ for $k \geq 0$. This links $c_0, c_2, c_4, \dots$ together and also $c_1, c_3, c_5, \dots$ We <u>expect</u> two arbitrary constants since $y'' + y = 0$ is 2^{nd} order,

$$\left.\begin{aligned}
c_2 &= -c_0/2 = \frac{-c_0}{2!} \\
c_3 &= -c_1/3\cdot 2 = \frac{-c_1}{3!} \\
c_4 &= -c_2/4\cdot 3 = \frac{c_0}{4\cdot 3\cdot 2} = \frac{c_0}{4!} \\
c_5 &= -c_3/5\cdot 4 = \frac{c_1}{5\cdot 4\cdot 3\cdot 2} = \frac{c_1}{5!} \\
c_6 &= -c_4/6\cdot 5 = \frac{-c_0}{6!}
\end{aligned}\right\}$$

we find the following pattern,

$$c_{2j} = \frac{(-1)^j c_0}{(2j)!} \quad (\text{even } c_n)$$

and,

$$c_{2j+1} = \frac{(-1)^j c_1}{(2j+1)!} \quad (\text{odd } c_n)$$

Thus the general sol$^{\underline{n}}$ can be written,

$$\boxed{y = c_0 \underbrace{\left(\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} \right)}_{\cos(x)} + c_1 \underbrace{\left(\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j+1} \right)}_{\sin(x)}}$$

— (no surprise I hope) —

Remark: Solving nonhomogeneous problems is not much harder. The difference is that there will be a portion of the sol$^{\underline{n}}$ which does not depend on arbitrary constants. That portion is the particular sol$^{\underline{n}}$.

$$\boxed{\boxed{E161}\quad \text{Solve} \quad y'' + y = g(x), \quad g(x) = \begin{cases} \sin(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}}$$

Let $y = \sum_{n=0}^{\infty} c_n x^n$ then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Notice $\frac{\sin(x)}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) = 1 - \frac{x^2}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j}$

Thus,

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j}$$

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) c_{k+2} + c_k\right] x^k = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j}$$

We'll find the RHS couples to the even c_k but not the odd k,

odd $k = 2j+1$ for some $j \geq 0$

$$(2j+1+2)(2j+1+1) c_{2j+1+2} + c_{2j+1} = 0$$

$$c_{2j+3} = \frac{-c_{2j+1}}{(2j+3)(2j+2)} \quad \text{for } j \geq 0$$

Now $c_3 = \frac{-c_1}{3!}$ and $c_5 = \frac{-c_3}{5 \cdot 4} = \frac{c_1}{5!}$ etc... $\underline{c_{2j+1} = \frac{(-1)^j c_1}{(2j+1)!}}$.
(just like $\boxed{E160}$)

even $k = 2j$ for some $j \geq 0$

$$(2j+2)(2j+1) c_{2j+2} + c_{2j} = \frac{(-1)^j}{(2j+1)!} \quad \left(\text{equating coefficients of } x^{2j}\right)$$

$$\underline{c_{2j+2} = \frac{(-1)^j}{(2j+2)(2j+1)(2j+1)!} - \frac{c_{2j}}{(2j+2)(2j+1)}}.$$

Continued ↘

E161 Continued: study formula for c_{2j+2} just found,

$j=0$ $\quad c_2 = \dfrac{1}{(2)(1)} - \dfrac{c_0}{2!} \Rightarrow \underline{c_2 = \left(\frac{1}{2} - \frac{1}{2}c_0\right)}.$

$j=1$ $\quad c_4 = \dfrac{-1}{(4)(3)3!} - \dfrac{c_2}{4\cdot 3} = \dfrac{-1}{72} - \dfrac{1}{12}\left(\frac{1}{2} - \frac{1}{2}c_0\right)$

$$\Rightarrow \underline{c_4 = \frac{-4}{72} + \frac{1}{4!}c_0}.$$

$j=2$ $\quad C_6 = \dfrac{1}{6\cdot 5\cdot 5!} - \dfrac{C_4}{6\cdot 5} = \dfrac{1}{5\cdot 6!} - \dfrac{1}{6\cdot 5}\left(\dfrac{-4}{3\cdot 4!} + \dfrac{1}{4!}C_0\right)$

$$\Rightarrow C_6 = \frac{1}{6!}\left(\frac{1}{5} + \frac{4}{3}\right) - \frac{1}{6!}C_0$$

$$\Rightarrow \underline{C_6 = \frac{23}{(15)6!} - \frac{C_0}{6!}}.$$

I'm not sure how to find a general formula for the terms above without c_0 ($1/2$, $-4/72$, $23/15\cdot 6!$ etc...). We can deduce the sol$^{\underline{n}}$ $y = \sum_{n=0}^{\infty} c_n x^n$ is as follows:

$$y = c_0\left(\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}x^{2j}\right) + c_1\left(\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}x^{2j+1}\right) + \frac{1}{2}x^2 - \frac{4}{72}x^4 + \cdots$$

$$= \underbrace{c_0\cos(x) + c_1\sin(x)}_{y_h} + \underbrace{\frac{1}{2}x^2 - \frac{4}{72}x^4 + \frac{23}{15\cdot 6!}x^6 + \cdots}_{y_p}$$

We have calculated the first three nontrivial terms in the particular sol$^{\underline{n}}$. If we want higher order terms we can find them by examining $j = 3, 4, \ldots$ I can't see a nice formula for the general coefficient of the terms in y_p.