THEORY FOR POWER SERIES SOLUTIONS : (§ 8.3 - 8.4)

Reminder: a function $f(x)$ is analytic at $x_0 \in \mathrm{dom}(f)$ if f has a power series representation $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for all x in some open interval around $x = x_0$. The distance from x_0 to the edge of the I.O.C. is called the Radius of Convergence (R.O.C.)

FACT: If f, g are analytic at $x = x_0$ then $f+g$, $f-g$, cf, fg and f/g (with $g(x_0) \neq 0$) are all also analytic at $x = x_0$.

We now turn to the question: "When does a linear ODE $L[y]=0$ have a series sol^{n} at $x = x_0$?". We consider the $n=2$ case primarily, but the principles and techniques are far more general.

Def^{n}/ Given the DEq^{n} (I) $a_2(x)\, y''(x) + a_1(x)\, y'(x) + a_0(x)\, y(x) = 0$ we call $y'' + py' + qy = 0$ it's <u>standard form</u>. A point x_0 is an <u>ordinary point</u> of (I) if $p = a_1/a_2$ and $q = a_0/a_2$ are <u>both</u> analytic at x_0. Otherwise x_0 is said to be a <u>singular point</u> of the DEq^{n} (I)

E162 $x^2 y'' + x y' + y = 0$ has standard form $y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0$ we identify that $P(x) = 1/x$ and $q(x) = \frac{1}{x^2}$. We find this Cauchy-Euler problem has the singular point $x_0 = 0$.

E163 $ay'' + by' + cy = 0$ has standard form $y'' + \left(\frac{b}{a}\right) y' + \left(\frac{c}{a}\right) y = 0$ if $a \neq 0$ clearly $P(x) = b/a$ and $q(x) = c/a$ are <u>constant</u> functions which are analytic over <u>all</u> of $\mathbb{R}$. It follows that $ay'' + by' + cy = 0$ has <u>no</u> singular points.

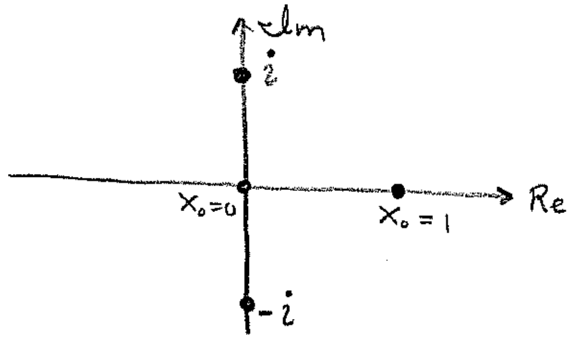
<u>Observation</u>: The sol^{n}s to the Cauchy-Euler problem do not include zero in their domain. On the other hand, the domain of the sol^{n}'s to $ay'' + by' + cy = 0$ is <u>all</u> of $\mathbb{R}$. This suggests that singular pt.

Th^{m}/ The DEq^{n} $y'' + py' + qy = 0$ has two linearly independent of the form $y(x) = \sum_{n=0}^{\infty} a_n (x-x_o)$ at each <u>ordinary point</u> x_o. Moreover, the radius of convergence is <u>at least</u> as large as the distance to the nearest singular pt. (we do consider complex values for x_o and "distance" is understood as usual in the complex plane.)

Pf: complex variables will make this less mysterious but you'll have to find a more advanced text for a proof of Th^{m} (5) (pg. 477)

E164 Find minimum R.O.C. for power series sol^{n} of $(x^2+1)y'' + 5y' + 6y = 0$ assuming the sol^{n} is centered at $x_o = 0$. What if $x_o = 1$?

In standard form the given DEq^{n} becomes $y'' + \frac{5}{x^2+1}y' + \frac{6}{x^2+1}y = 0$. We identify $P(x) = \frac{5}{x^2+1}$ and $q(x) = \frac{6}{x^2+1}$. It follows $\nexists$ any <u>real</u> singular point $\Rightarrow$ we can find analytic sol^{n} at each point in $\mathbb{R}$. Consider $x \in \mathbb{C}$, any quadratic factors over $\mathbb{C}$ notice that $x^2+1 = (x+i)(x-i) \Rightarrow x = \pm i$ are <u>complex</u> singular pts.



looking at the complex plane it's clear that $x = \pm i$ are the same distance from $x_o = 1$, namely $\sqrt{1^2+1^2} = \sqrt{2}$. On the other hand $x_o = 0$ is distance 1 from the singular pts.

By Th^{m} (5),

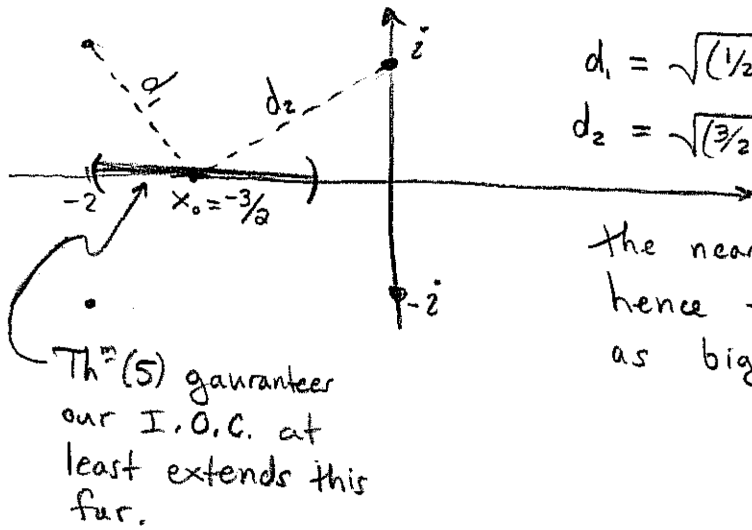
$$\boxed{\begin{array}{l} x_o = 0 \Rightarrow \text{R.O.C.} \geq 1 \\ x_o = 1 \Rightarrow \text{R.O.C.} \geq \sqrt{2} \end{array}}$$

E165 Find complex singular pts. of $(x^2+4x+5)(x^2+1)y'' + (x^2+1)y' + y = 0$

Identify $P(x) = \frac{1}{x^2+4x+5}$ and $q(x) = \frac{1}{(x^2+4x+5)(x^2+1)}$. If either P or q has division by zero we get singular pt; $\boxed{x = \pm i \text{ and } x = -2 \pm i}$

$\boxed{E166}$ We see from $\boxed{E165}$ that $x(x^2+4x+5)(x^2+1)y'' + (x^2+1)y' + y = 0$ has only $x=0$ as real sing. points, however $x = \pm i$ and $x = -2 \pm i$ are complex singular points. Find minimum R.O.C for sol^{n} centered at $x_o = -3/2$ $\left(y = \sum_{n=0}^{\infty} c_n (x+\tfrac{3}{2})^n \right)$

We plot all the singular points on the complex plane and use Th^{m}(5),

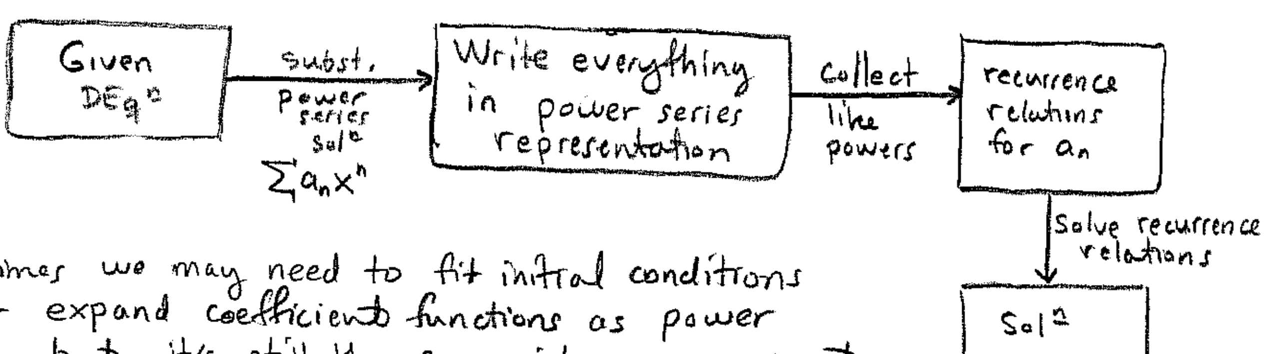


$$d_1 = \sqrt{(1/2)^2 + (1)^2} = \sqrt{5/4} = \tfrac{1}{2}\sqrt{5}$$
$$d_2 = \sqrt{(3/2)^2 + (1)^2} = \sqrt{13/4} = \tfrac{1}{2}\sqrt{13}$$

the nearest singular pts are at $-2 \pm i$ hence the R.O.C. is at least as big as $\sqrt{5}/2$

Remark: pgs. ~~76~~-80 of Penrose's "THE ROAD TO REALITY; A COMPLETE GUIDE TO THE LAWS OF THE UNIVERSE" discuss at length the difference between the functions $f(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{1}{1-x^2}$ in view of power series and complex singularities. Incidentally PENROSE's book is a treasure of usual & unusual intuitive mathematics.

Remark: by now I've spent entirely too much time on how to locate and analyze singular points. Anyway, we've seen about all there is to see for §8.2-8.4. The overall idea is relatively simple.



Sometimes we may need to fit initial conditions and/or expand coefficient functions as power series but it's still the same idea: represent everything in power series and then calculate.

E167 Find terms up to order 2 in power series sol^{n} centered at $x_0=0$ for $\tan^{-1}(x)\, y'' + e^x y' + 2y = 0$.

As usual we assume $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$

Recall $\tan^{-1}(x) = g(x) \Rightarrow \frac{dg}{dx} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$

thus $\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots$

and we know $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$

Substitute all these into the given DEq^{n},

$$\left(x - \tfrac{1}{3}x^3\right)\left(2c_2 + 6c_3x + 12c_4x^2\right) + \left(1 + x + \tfrac{x^2}{2}\right)\left(c_1 + 2c_2x + 3c_3x^2\right) + \cdots$$
$$+\, 2\left(c_0 + c_1x + c_2x^2\right) + \cdots = 0$$

I intend to keep all terms which can yield terms of order less than or equal to two (x^2, x or constant-type terms). When I multiply I only look for terms of order 2 or less, this simplifies the multiplication greatly,

$$2c_2x + 6c_3x^2 + \cdots + c_1 + 2c_2x + 3c_3x^2 + c_1x + 2c_2x^2 + \tfrac{1}{2}c_1x^2 + \cdots$$
$$+\, 2c_0 + 2c_1x + 2c_2x^2 + \cdots = 0$$

Collect like powers,

$$(c_1 + 2c_0) + (2c_2 + 2c_2 + c_1 + 2c_1)x + \left(6c_3 + 3c_3 + 2c_2 + \tfrac{c_1}{2} + 2c_2\right)x^2 + \cdots = 0$$

It follows we wish to solve,

$$c_1 + 2c_0 = 0 \Rightarrow \underline{c_1 = -2c_0}.$$
$$3c_1 + 4c_2 = 0 \Rightarrow \underline{c_2 = -\tfrac{3}{4}c_1 = \tfrac{3}{2}c_0}.$$
$$\tfrac{1}{2}c_1 + 4c_2 + 9c_3 = 0 \Rightarrow c_3 = \left(\tfrac{-1}{2}c_1 - 4c_2\right)/9 = (c_0 - 6c_0)/9 = \tfrac{-5}{9}c_0.$$

Curious, we find the DEq^{n} examined to quadratic order reveals coeff. of x^3,

$$\underline{y = c_0\left(1 - 2x + \tfrac{3}{2}x^2 - \tfrac{5}{9}x^3 + \cdots\right)}.$$

Even more curious, where is our 2nd linearly independent sol^{n}?

$\boxed{\text{FIND MY ERROR}}$

$\boxed{E168}$ Solve $(x+3)y'' + x^2y' + y = 0$ upto order 2 in x

Let $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$

$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots$

$y'' = 2c_2 + 6c_3x + 12c_4x^2 + \cdots$

Substitute, keep only terms x^2 or lower,

$$(x+3)(2c_2 + 6c_3x + 12c_4x^2 + \cdots) + x^2(c_1 + 2c_2x + \cdots) + c_0 + c_1x + c_2x^2 + \cdots = 0$$

$$(6c_2 + c_0) + (2c_2 + 18c_3 + c_1)x + (6c_3 + 36c_4 + c_1 + c_2)x^2 + \cdots = 0$$

It follows,

$$6c_2 + c_0 = 0 \longrightarrow \underline{c_2 = \frac{-1}{6}c_0}.$$

$$2c_2 + 18c_3 + c_1 = 0 \longrightarrow c_3 = \frac{-1}{18}(2c_2 + c_1) = \frac{-1}{18}\left(\frac{-c_0}{3} + c_1\right) = \underline{\frac{c_0}{54} - \frac{1}{18}c_1}.$$

$$6c_3 + 36c_4 + c_1 + c_2 = 0$$

$$c_4 = \frac{-1}{36}(6c_3 + c_1 + c_2) = \frac{-1}{36}\left(6\left(\frac{c_0}{54} - \frac{c_1}{18}\right) + c_1 - \frac{c_0}{6}\right)$$

$$c_4 = \frac{-1}{36}\left(\frac{2}{3}c_1 + \left(\frac{1}{9} - \frac{1}{6}\right)c_0\right) = \underline{\frac{-c_1}{54} - \frac{1}{36}\left(\frac{-3}{54}\right)c_0}.$$

Consequently we identify c_0, c_1 as the arbitrary constants of the general sol$^{\underline{n}}$ and, using our results above,

$$\boxed{y = c_0\left(1 - \frac{1}{6}x^2 + \frac{1}{54}x^3 + \frac{1}{12(54)}x^4 + \cdots\right) + c_1\left(x - \frac{1}{18}x^3 - \frac{1}{54}x^4 + \cdots\right)}$$

Reminder: go over Example 3 on pg. 479 of text. — (I don't have one like that in these notes)

- There are additional examples in the Practice Homework Sol$^{\underline{n}}$s.

- We now turn to the question of what to do with singular points, we'll need something beyond power series. The Method of Frobenius will serve to attack a large class of problems currently inaccessible to us with our techniques of §8.1 → 8.4.