Method of Frobenius : §8.6 — 8.7 (developed in 1873)

We have seen sol^{n}'s of the form $c_0 + c_1 x + c_2 x^2 + \cdots$ cover a great variety of problems. The Frobenius Method extends the guess to $x^r(c_0 + c_1 x + c_2 x^2 + \cdots)$. The proposed sol^{n} will not always be a power series, for example if $r = 1/2$ then we'd have a sol^{n} of the form $\sqrt{x}(c_0 + c_1 x + \cdots) = c_0\sqrt{x} + c_1 x^{3/2} \cdots$

It turns out we can describe sol^{n}'s of DEq^{n}'s with a particular type of singular point.

Def^{n}/ A singular point x_0 of $y''(x) + P(x)y'(x) + q(x)y(x) = 0$ is said to be a regular singular point if both $(x - x_0)P(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0. Otherwise x_0 is called an irregular singular point

In complex variables $f(z)$ is said to have a pole of order m at z_0 if $f(z)(z - z_0)^m$ is analytic but $f(z)(z - z_0)^{m-1}$ is not analytic. So we could say regular singular points are those of order 1 or 2 for q and/or P. An essential singularity is one for which no m makes $f(z)(z - z_0)^m$ analytic, $\exp\left(\frac{1}{z}\right)$ as essential singularity at zero.

Digression, unless you're taking complex variables some time soon.

$\boxed{\text{E169}}$ $x^2 y'' + xy' + y = 0$ (Cauchy Euler DEq^{n})

$$\Rightarrow y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0$$

$$\Rightarrow P(x) = \frac{1}{x} \text{ and } q(x) = \frac{1}{x^2}$$

Observe $xP(x) = 1$ and $x^2 q(x) = 1$ thus xP and $x^2 q$ are both analytic at the singular point $x_0 = 0$, hence x_0 is a regular singular point.

Remark: Cauchy Euler problems $ax^2y'' + bxy' + cy = 0$ have regular singular points at $x_o = 0$. Recall we found that the sol^n's of the form $y = x^r$ or $y = \ln(x)x^r$ for real roots r of the Characteristic or Indicial Eq^n (see 94 – 95 or §8.5). This means the Cauchy Euler sol^n's are a very special case of the Frob. Sol^n $y = x^r(c_o + c_1x + c_2x^2 + \cdots)$ It's the case where $0 = c_1 = c_2 = \cdots$. In section 8.7 we'll learn $y = \ln(x)x^r$ is also a special case of the repeated root case for the Frob. Sol^n. Let me turn this around:

- **Frobenius Method:** DEq^n's with a regular singular point x_o are essential a slight twist of a Cauchy Euler Problem thus we suspect a sol^n of the form $(y_{\text{Cauchy Euler}})(\text{analytic function})$ will solve it. This is the intuition of the method roughly speaking.

Let me provide a few supporting calculations (roughly pg. 489 of text)

Def^n/ The <u>Frobenius Sol^n</u> of index r is

$$W(r,x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

In general r could be complex, we've discussed how that works for Cauchy Euler problems however the text limits discussion to real case in §8.6 – 8.7 (Problem #26 of §8.7 explores complex case.)

THE Problem: Solve $y'' + py' + qy = 0$ at a REGULAR SINGULAR POINT x_o

Observation: this is like Cauchy-Euler Problem try the "Frobenius Sol^{n}" $y = \sum_{n=0}^{\infty} a_n (x - x_o)^{n+r}$. Since we can always shift x_o to zero we can let $x_o = 0$ without losing any generality,

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Completely unjustified if you want to get technical these are not power series so we have no Th^{m}'s for such calculation. You'll need to consult a more sophisticated source for proofs.

Now substitute into the given DEq^{n},

$$(*) \quad \boxed{\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(\sum_{n=0}^{\infty} P_n x^{n-1}\right)\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} q_n x^{n-2}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0}$$

likewise, not justified but on the other hand not surprising really.

Notice we assumed $x_o = 0$ is regular singular point hence $xP(x)$ and $x^2 q(x)$ are analytic, it follows that $P(x) = \frac{P_0}{x} + P_1 + P_2 x^2 + \cdots$ and $q(x) = \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3 x + \cdots$

Since the singular pt. $x_o = 0$ is regular we know at least one of P_0, q_0 is non zero. These are Laurent Expansions, again complex variables discusses these at length.

<u>Continuing</u>:

We need to analyze (*). What does it say about the coefficients of our Frobenius Sol$^{\underline{n}}$? Let's change notation to $+\cdots$ to calculate a few of the lowest order terms, (we assume $a_0 \neq 0$)

$$r(r-1)a_0x^{r-2} + p_0\frac{1}{x}\left(ra_0x^{r-1}\right) + q_0x^{-2}\left(a_0x^r\right) + \quad \left(\begin{matrix}x^{r-2}\\ \text{terms}\end{matrix}\right)$$

$$+ (1+r)ra_1x^{r-1} + p_0\frac{1}{x}\left((1+r)a_1x^r\right) + p_1\left(ra_0x^{r-1}\right) +$$

$$+ \left(q_0\frac{1}{x^2} + q_1\frac{1}{x}\right)\left(a_0x^r + a_1x^{1+r}\right) + \cdots = 0 \quad \left(\begin{matrix}x^{r-1}\\ \text{power}\\ \text{terms}\end{matrix}\right)$$

$$x^{r-2}\Big(a_0\big(r(r-1) + p_0r + q_0\big)\Big) +$$

$$+ x^{r-1}\Big(a_1r(r+1) + a_1p_0(r+1) + p_1a_0r + q_0a_1 + q_1a_0\Big) + \cdots = 0$$

The coefficients must vanish thus as $a_0 \neq 0$ we find $r(r-1) + p_0r + q_0 = 0$ from the x^{r-2} term. This is called the <u>indicial eq$^{\underline{n}}$</u>.

Def$^{\underline{n}}$/ $r(r-1) + p_0r + q_0 = 0$ indicial eq$^{\underline{n}}$ for x_0

where $p_0 = \lim_{x\to x_0}(x-x_0)p(x)$, $q_0 = \lim_{x\to x_0}(x-x_0)^2q(x)$

we call <u>roots</u> of the indicial eq$^{\underline{n}}$ the <u>exponents</u> of the singularity x_0 of the eq$^{\underline{n}}$ $y'' + py' + qy = 0$

Notice the differential eq$^{\underline{n}}$ $y'' + p(x)y' + q(x)y = 0$ is approximately $y'' + p_0y' + q_0y = 0$ for $x \to x_0$ since $p(x) = \frac{p_0}{x-x_0} + \cdots$ and $q(x) = \frac{q_0}{(x-x_0)^2} + \cdots$ are dominated by the singular pieces near the singularity.

FROBENIUS METHOD: (I combine §8.6–8.7 for whole story)

Given $y''+py'+qy=0$ with regular singular point x_0,

(1.) Calculate the indicial eq^{n} $r(r-1)+P_0 r+q_0=0$ and solve to find the exponents which solve the quadratic eq^{c}, call them r_1, r_2 with $Re(r_1)\geq Re(r_2)$.

(2.) Assume $W(r_1,x)$ solves $y''+py'+qy=0$. Substitute and find coefficients of $W(r,x)=a_0x^{r_1}+a_1x^{r_1+1}+\cdots$ as requested (sometimes asked to solve general recurrence relations that result, other times I just ask for 1st few terms). Let $W(r_1,x)=y_1$

(3.) If $r_1=r_2$ then $y_2(x)=y_1(x)\ln(x-x_0)+\sum_{n=1}^{\infty}b_n(x-x_0)^{n+r}$ this y_2 will be linearly independent from y_1 (note there is much flexibility in choice of b_n above)

(4.) If $r_1-r_2\notin\mathbb{Z}$ (r_1-r_2 is not an integer) then $W(r_2,x)=b_0x^{r_2}+b_1x^{r_2+1}+\cdots$ is a sol^{n} and you can find b_n as needed. Again $y_2(x)=W(r_2,x)$ is linearly independent from y_1 we found in (2.).

(5.) If $r_1-r_2\in\mathbb{Z}$ then a 2nd linearly independent sol^{n} of the form that follows is linearly independent of y_1,

$$y_2(x)=cy_1(x)\ln(x-x_0)+\sum_{n=0}^{\infty}b_n(x-x_0)^{n+r_2},\quad b_0\neq 0$$

where c is a constant that could be zero.

(6.) If the exponents are complex consult other texts. (or try to extrapolate from #26 of §8.7)

The general sol^{n} to $y''+py'+qy=0$ is $y=c_1y_1+c_2y_2$ and it's I.O.C. extends at least as far the next nearest singular point. (Th^{m} 6 of §8.6)

Remark: To summarize, the Method of Frobenius says we can solve a DEq$^{\underline{n}}$ at a regular singular pt. with a sol$^{\underline{n}}$ of the form $y = a_0 x^r + a_1 x^{r+1} + \dots$ and the value of r is selected from the indicial eq$^{\underline{n}}$.

E170 Find first 6 terms in general sol$^{\underline{n}}$ of the differential equation $y'' + xy' - \frac{2}{x^2}y = 0$

(taken from Oneil's Advanced Engineering Mathematics Problem #13 §5.3)

Observe $P(x) = x$ and $q(x) = \frac{-2}{x^2}$. Comparing these formulas to $P(x) = \frac{P_0}{x} + P_1 + P_2 x + \cdots$ and $q(x) = \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3 x + \cdots$ to the given functions reveals $P_2 = 1$ and all other $P_k = 0$ whereas $q_0 = -2$ and $q_k = 0$ for $k \neq 0$. The indicial eq$^{\underline{n}}$ is,

$$r(r-1) - 2 = 0 \Rightarrow r^2 - r - 2 = 0$$
$$\Rightarrow (r-2)(r+1) = 0$$
$$\Rightarrow \underline{r_1 = 2 \text{ and } r_2 = -1}.$$

We seek to determine coeff. in $W(2,x) = a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots = y_1$

$$y_1'' + xy_1' - \frac{2}{x^2}y_1 = 0$$
$$2a_0 + 6a_1 x + 12a_2 x^2 + x(2a_0 x) - \frac{2}{x^2}(a_0 x^2 + a_1 x^3 + a_2 x^4) + \cdots = 0$$
$$\underbrace{(2a_0 - 2a_0)}_{\text{nice, no condition on } a_0.} + x\underbrace{(6a_1 - 2a_1)}_{\text{yields } 4a_1 = 0} + x^2\underbrace{(12a_2 + 2a_0 - 2a_2)}_{10a_2 = -2a_0,\ a_2 = \frac{-1}{5}a_0} + \cdots = 0$$

This gives us first two nontrivial terms in y_1, set $a_0 = 1$

$$\boxed{y_1(x) = x^2 - \frac{1}{5}x^4 + \cdots}$$

If we did additional calculation we'd find

$$\underline{y_1(x) = x^2 - \frac{1}{5}x^4 + \frac{1}{5(7)}x^6 - \frac{1}{5(7)(9)}x^8 + \frac{1}{5(7)(9)(11)}x^{10} + \cdots}.$$

(Math 501 at NCSU has hwk. problems like this)

E170 Continued:

We wish to find the 2^{nd} L.I. sol^n $y_2(x)$. Notice that $r_2 - r_1 = -3 \in \mathbb{Z}$ hence we seek sol^n of form,

$$y_2(x) = c y_1(x)\ln(x) + b_0 x^{-1} + b_1 + b_2 x + b_3 x^2 + \cdots \quad \text{I}$$

$$y_2' = c y_1' \ln(x) + c y_1/x - \frac{b_0}{x^2} + b_2 + 2b_3 x + 3b_4 x^2 + 4b_5 x^3 + \cdots$$

$$x y_2' = c\ln(x)\, x y_1' + c y_1 - b_0/x + b_2 x + 2b_3 x^2 + \cdots \quad \text{II}$$

$$y_2'' = c y_1'' \ln(x) + c y_1'/x + c y_1'/x - c y_1/x^2 + \frac{2b_0}{x^3} + 2b_3 + 6b_4 x + 12 b_5 x^2 + \cdots \quad \text{III}$$

Substitute I, II and III into $y'' + xy' - \frac{2}{x^2}y = 0$

$$c\ln(x)\left(y_1'' + x y_1' - \frac{2}{x^2}y_1\right) + 2c y_1'/x - c y_1/x^2 + \frac{2b_0}{x^3} + 2b_3 + 6b_4 x +$$
$$+ 12 b_5 x^2 + c y_1 - b_0/x + b_2 x + 2b_3 x^2 - \frac{2b_0}{x^3} - \frac{2b_1}{x^2} - \frac{2b_2}{x} - 2b_3 -$$
$$- 2b_4 x - 2b_5 x^2 + \cdots = 0$$

Now substitute in $y_1 = x^2 - \frac{1}{5}x^4 + \cdots$ and $y_1' = 2x - \frac{4}{5}x^3 + \cdots$. Observe also $y_1'' + x y_1' - \frac{2}{x^2}y_1 = 0$ so the 1^{st} term vanishes,

$$2c\frac{1}{x}\left(2x - \frac{4}{5}x^3 + \cdots\right) - c\frac{1}{x^2}\left(x^2 - \frac{1}{5}x^4 + \cdots\right) +$$
$$+ \cancel{\frac{2b_0}{x^3}} + 2b_3 + 6b_4 x + 12 b_5 x^2 + c\left(x^2 - \frac{1}{5}x^4\right) -$$
$$- \frac{b_0}{x} + b_2 x + 2b_3 x^2 - \cancel{\frac{2b_0}{x^3}} - \frac{2b_1}{x^2} - \frac{2b_2}{x} - 2b_3 - 2b_4 x - 2b_5 x^2 + \cdots = 0$$

Can ignore next stage since it's beyond x^2

I'll keep terms upto order x^2,

$$\boxed{\begin{aligned}&(4c - c + 2b_3 - 2b_3) + x(6b_4 + b_2 - 2b_4) + x^2\left(-\frac{8c}{5} + \frac{c}{5} + 12b_5 + c + 2b_3 - 2b_5\right)\\ &+ (-b_0 - 2b_2)\frac{1}{x} + (-2b_1)\frac{1}{x^2} + \cdots = 0\end{aligned}}$$

$\boxed{2}$

E170 Continued

From $\boxed{C}$ we find,

$$\left.\begin{array}{ll} \underline{\text{Const}} & 3c = 0 \\ \underline{x^1} & 4b_4 + b_2 = 0 \\ \underline{x^2} & 10b_5 + 2b_3 = 0 \\ \underline{\frac{1}{x}} & -b_0 - 2b_2 = 0 \\ \underline{\frac{1}{x^2}} & -2b_1 = 0 \end{array}\right\} \quad \begin{array}{l} \underline{c = 0}. \\ b_4 = -\frac{1}{4}b_2 \\ b_5 = \frac{-1}{5}b_3 \\ \underline{b_2 = \frac{-1}{2}b_0}. \Rightarrow \underline{b_4 = \frac{1}{8}b_0}. \\ \underline{b_1 = 0}. \end{array}$$

Notice that b_3 and b_5 go with x^2 and x^4 in the sol$^{\underline{n}}$; I anticipate $b_3(x^2 - \frac{1}{5}x^4 + \cdots)$ is just another copy of y_1 which appears (in principle) in y_2. For convenience we set $b_3 = 0$ and $b_0 = 1$ to find, (using I)

$$\boxed{y_2(x) = \frac{1}{x} - \frac{1}{2}x + \frac{1}{8}x^3 + \cdots}$$

Thus the general sol$^{\underline{n}}$ $y = c_1 y_1 + c_2 y_2$ has the explicit form

$$\boxed{y = c_1\left(x^2 - \frac{1}{5}x^4 + \cdots\right) + c_2\left(\frac{1}{x} - \frac{1}{2}x + \frac{1}{8}x^3 + \cdots\right)}$$

If you think about it you'll see that a non zero choice of b_1 amounts to redefining c_1 in the general sol$^{\underline{n}}$. This ambiguity is likely to present itself when we work problems that fall into case (5.) as described on 175.

Incidentally, if we calculated more terms we'd find

$$\underline{y_2(x) = \frac{1}{x} - \frac{1}{2}x + \frac{1}{8}x^3 - \frac{1}{48}x^5 + \frac{1}{384}x^7 - \frac{1}{3840}x^9}$$