CASE STUDY IN SYMMETRY METHODS

Def^n/ A symmetry of the differential equation $\frac{dy}{dx} = \omega(x,y)$ is a smooth transformation $\Gamma_\varepsilon(x,y) = (\bar{x},\bar{y})$ such that $\frac{d\bar{y}}{d\bar{x}} = \omega(\bar{x},\bar{y})$. Cannonical coordinates (r,s) are coordinates such that $\Gamma_\varepsilon(r,s) = (r, s+\varepsilon)$.

E32 This is Example 1.3 from Peter Hydon's "Symmetries in DEq^n's : A BEGINNERS GUIDE". Consider:

$$\boxed{\frac{dy}{dx} = \frac{y^3 + x^2 y - y - x}{xy^2 + x^3 + y - x}} \quad (*)$$

A rotation by angle ε has the form (see my Appendix on Complex Math)

$$\bar{x} = x\cos\varepsilon - y\sin\varepsilon$$
$$\bar{y} = x\sin\varepsilon + y\cos\varepsilon$$

Let's see if this transformation is a symmetry of $(*)$, it's a little tedious in Cartesian Coordinates (ask me I'll show you). I'll change to polar coordinates ; $x = r\cos\theta$, $y = r\sin\theta$. Note $(*)$ can be written as:

$$\big(x(y^2+x^2) + y - x\big)dy - \big(y(y^2+x^2) - y - x\big)dx = 0$$

Note $dx = \cos\theta\, dr - r\sin\theta\, d\theta$ and $dy = \sin\theta\, dr + r\cos\theta\, d\theta$ thus,

$$\big(r^3\cos\theta + r\sin\theta - r\cos\theta\big)\big(\sin\theta\, dr + r\cos\theta\, d\theta\big) +$$
$$-\big(r^3\sin\theta - r\sin\theta - r\cos\theta\big)\big(\cos\theta\, dr - r\sin\theta\, d\theta\big) = 0$$

$$\big[\cancel{r^3\sin\theta\cos\theta} + r(\sin^2\theta - \cancel{\sin\theta\cos\theta}) - \cancel{r^3\sin\theta\cos\theta} + r(\cancel{\sin\theta\cos\theta} + \cos^2\theta)\big]dr +$$
$$+\big[r^4\cos^2\theta + r^2(\cancel{\sin\theta\cos\theta} - \cos^2\theta) + r^4\sin^2\theta - r^2(\sin^2\theta + \cancel{\sin\theta\cos\theta})\big]d\theta = 0$$

$$r\,dr + (r^4 - r^2)\,d\theta = 0 \longrightarrow \underbrace{\frac{dr}{d\theta} = \frac{r^4 - r^2}{r} = r^3 - r}_{(*)\text{ in polar coordinates}}$$

The rotation in polar coordinates simplifies considerably,

$$\bar{r} = r$$
$$\bar{\theta} = \theta + \varepsilon$$

(polar coordinates are cannonical coordinates for this Diff. Eq^n)

It is clear that $\frac{d\bar{r}}{d\bar{\theta}} = \bar{r}^3 - \bar{r}$ given that $\frac{dr}{d\theta} = r^3 - r$.

Continued

E32 Continued

Notice that in polar coordinates $(*)$ takes a particularly nice form, it's separable

$$\frac{dr}{d\theta} = r^3 - r$$

$$\frac{dr}{r^3 - r} = d\theta$$

Partial fractions, $\frac{1}{r^3-r} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r-1}$ multiplying by $r^3 - r$ yields $1 = A(r^2-1) + Br(r-1) + Cr(r+1)$. Substitute roots $r = 0, \pm 1$ to find $1 = -A$, $1 = 2C$, $1 = 2B$ thus,

$$\int \frac{dr}{r^3 - r} = \int \left(\frac{-1}{r} + \frac{1/2}{r+1} - \frac{1/2}{r-1} \right) dr$$

$$\Rightarrow \quad \theta = -\ln|r| + \frac{1}{2}\ln|r+1| - \frac{1}{2}\ln|r-1| + C$$

Now return to Cartesian coordinates,

$$\boxed{\tan^{-1}(y/x) = \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right) + \ln\sqrt{\frac{\sqrt{x^2+y^2}+1}{\sqrt{x^2+y^2}-1}} + C}$$

This is the general idea:

1. find symmetry for given DEq$^{\underline{n}}$.
2. find cannonical coordinates for the symmetry.
3. change differential eq$^{\underline{n}}$ to cannonical coordinates.
4. solve the separable DEq$^{\underline{n}}$
5. convert back to Cartesian coordinates.

It would take dozens of pages to illustrate this properly. Peter Hydon's Text gives a better idea about what a "symmetry" entails. In fact, there is some sort of symmetry behind most of the substitutions and/or $\int$-factors we've discussed. In other words, symmetry is a unifying theme throughout the mathematics of Chapter 2. continued

Continuing

However, the symmetry is hidden and not emphasized by the text or me most places. In applications, especially physics, the symmetry plays a central role. In fact, it is customary to <u>begin</u> the discussion of many physical problems with a proper choice of <u>coordinates</u>. We use <u>spherical</u> coordinates for a problem with <u>spherical</u> <u>symmetry</u>. We use <u>cylindrical</u> coordinates for a problem with rotational symmetry about a central axis. If motion is along some tilted plane sometimes we use coordinates based on the motion. In physic's, much simplification is gained by a proper choice of coordinates. Essentially this is the same idea as the <u>cannonical</u> <u>coordinates</u> defined for our $\boxed{E32}$.

- Another interesting application of symmetries is in generating the general $sol^{\underline{n}}$. Given a specific $sol^{\underline{n}}$ if we translate it by the symmetry then <u>usually</u> we'll obtain a new $sol^{\underline{n}}$.

<u>Remark</u>: Symmetry of Differential Equations is an active and interesting area of math research. It involves both abstract algebra & analysis, moreover much research demands some sophistication with modern mathematical software (Mathematica, Maple, Matlab etc...)