Newtonian Mechanics: §3.4

You may recall Newton's Laws of motion, paraphrasing a bit,

1.) A body subject to no external force has constant velocity
2.) A body subject to an external force has a time rate of change of momenta equal to the vector sum of the external force
3.) Equal and opposite reaction

Now I prefer,

1.) $\vec{F}_{ext} = 0 \Rightarrow \frac{d}{dt}(\vec{V}) = 0$

2.) $\vec{F}_{ext} = \frac{d\vec{P}}{dt}$

3.) $\vec{F}_{12} = -\vec{F}_{21}$ F_{12} is force of 1 on 2, F_{21} is force of 2 on 1

Where $\vec{F}_{ext}$ is the net external force on the object and $\vec{P} = m\vec{V}$ is the momentum of the object. Usually 2.) looks like

$$F = \frac{d}{dt}(mV) = \frac{dm}{dt}V + m\frac{dV}{dt} \Rightarrow F = ma$$

$$\left(\text{when } \frac{dm}{dt} = 0\right)$$

To begin with we say $F = ma$ but in general $F = \frac{dP}{dt}$ says much more.

<u>Remark</u>: See pg. 174 $\boxed{E6}$ for an example of a problem with $\frac{dm}{dt} \neq 0$ (under the Calculus II notes)

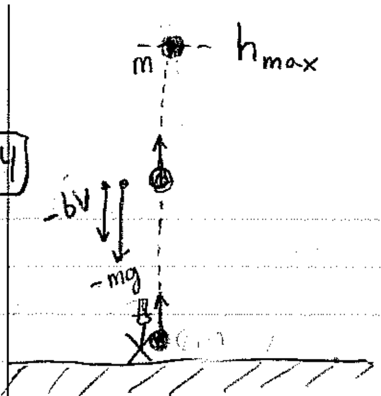
$\boxed{E33}$ Suppose a ball is rolled on a track with friction proportional to the velocity. Suppose β is the proportionality constant and it's intial velocity is V_0,

$$F = ma = -\beta V \Rightarrow m\frac{dV}{dt} = -\beta V$$

$$\Rightarrow \int \frac{dV}{V} = \int \frac{-\beta}{m}\, dt$$

$$\ln|V| = \frac{-\beta}{m}t + C_1 \Rightarrow \boxed{V = V_0 e^{-\frac{\beta}{m}t}}$$

the velocity slows exponentially to zero.



E34

imagine a ball is thrown vertically against the force of gravity. Assume friction $F_f = -bV$ for $b > 0$ and $F_{gravity} = -mg$.

I take $y = 0$ to be ground level and y increases as you go up. Newton's 2nd law yields,

$$m\frac{dV}{dt} = -mg - bV \quad , \quad V = \frac{dy}{dt}.$$

$$\frac{dV}{dt} + \frac{b}{m}V = -g \Rightarrow \mu = e^{\frac{b}{m}t} \quad \text{thus,}$$

$$\frac{d}{dt}\left(e^{\frac{bt}{m}}V\right) = -ge^{\frac{b}{m}t} \Rightarrow V(t) = c_1 e^{-\frac{bt}{m}} - gm/b$$

Call the intial velocity V_o $\Rightarrow$ $\boxed{V(t) = \left(V_o + gm/b\right)e^{-\frac{bt}{m}} - gm/b}$

When $b = 0$ it's easier, $\frac{dV}{dt} = -g \therefore V = V_o - gt$ and we can determine the time at which h_{max} is obtained by observing $V = 0$ at h_{max} so $\boxed{t = V_o/g}$ Let's compare that to the time it takes for the ball with friction to reach h_{max}.

$$0 = \left(V_o + gm/b\right)e^{-bt/m} - gm/b$$

$$\frac{gm/b}{V_o + gm/b} = e^{-bt/m} \Rightarrow \ln\left(\frac{gm/b}{V_o + gm/b}\right) = -\frac{bt}{m}$$

Thus, $t = -\frac{m}{b}\ln\left(\frac{gm/b}{V_o + gm/b}\right) \therefore \boxed{t = \frac{m}{b}\ln\left(1 + \frac{V_o b}{gm}\right)}$

Recall $\ln(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 + \dots$ let $u = V_o b/gm$

$$t = \frac{m}{b}\left(\frac{V_o b}{gm} - \frac{1}{2}\left(\frac{V_o b}{gm}\right)^2 + \dots\right) = \frac{V_o}{g} - \frac{1}{2}\frac{bV_o^2}{g^2 m} + \dots$$

We find the time to maximum height is shorter than the time $\left(\frac{V_o}{g}\right)$ to the zenith w/o friction

<u>Remark</u>: on the way back down $F_f = -b\frac{dy}{dt} > 0$ and $\vec{F_f}$ upward since $\frac{dy}{dt} < 0$ as y decreases.

E35

Let m, k be constants with $m > 0$. Suppose a force of $F = kv^3$ is applied to the mass m. Furthermore, suppose $v = \frac{dx}{dt}$ and $x(0) = 0$ while $v(0) = 10$. Find the equation of motion by solving Newton's Law. Also find velocity as fnct. of x.

$$m \frac{dv}{dt} = kv^3 \quad \& \quad x(0) = 0, \ v(0) = 10$$

Separate variables, integrate

$$\int \frac{dv}{v^3} = \int \frac{k}{m}\, dt \quad \Rightarrow \quad \frac{-1}{2v^2} = \frac{kt}{m} + C_1$$

Solve for v,

$$v = \pm \sqrt{\frac{-1}{2}\left(\frac{1}{kt/m + C_1}\right)}$$

Notice $v(0) = 10$ thus we must choose the $(+)$ possibility,

$$10 = \sqrt{\frac{-1}{2C_1}} \quad \Rightarrow \quad 100 = \frac{-1}{2C_1} \quad \Rightarrow \quad \underline{\underline{C_1 = \frac{-1}{200}}}$$

Thus, we find

$$\boxed{v(t) = \left(\frac{1}{100} - \frac{2kt}{m}\right)^{-1/2}}$$

To find position $x(t)$ we integrate $v(t) = \frac{dx}{dt}$ with respect to t,

$$\int_0^t \frac{dx}{d\bar{t}}\, d\bar{t} = \int_0^t v(\bar{t})\, d\bar{t} \qquad \left(\bar{t} \text{ is a dummy variable of integration}\right)$$

$$\| \, (\text{F.T.C.}) \qquad\qquad \|$$

$$x(t) - x(0) = \int_0^t \left(\frac{1}{100} - \frac{2k\bar{t}}{m}\right)^{-1/2} d\bar{t} \quad ; \quad u = \frac{1}{100} - \frac{2k\bar{t}}{m}$$

$$= \frac{-m}{k}\left(\frac{1}{100} - \frac{2k\bar{t}}{m}\right)^{1/2}\Big|_0^t$$

$$= \boxed{\frac{-m}{k}\left(\frac{1}{100} - \frac{2kt}{m}\right)^{1/2} + \frac{m}{10k} = x(t)}$$

On the next page I find the same result by a slightly twisted approach, The trick below is one of my favorites,

$$m\frac{dV}{dt} = kV^3$$

sneaky.

$$m\frac{dx}{dt}\frac{dV}{dx} = mV\frac{dV}{dx} = kV^3$$

$$\int \frac{m}{k}\frac{dV}{V^2} = \int dx$$

$$\frac{-m}{kV} + C_2 = X$$

Now when $t=0$ we have $\left.\begin{array}{l} X(t=0) = 0 \\ V(t=0) = 10 \end{array}\right\} \Rightarrow \underline{V(X=0) = 10}$

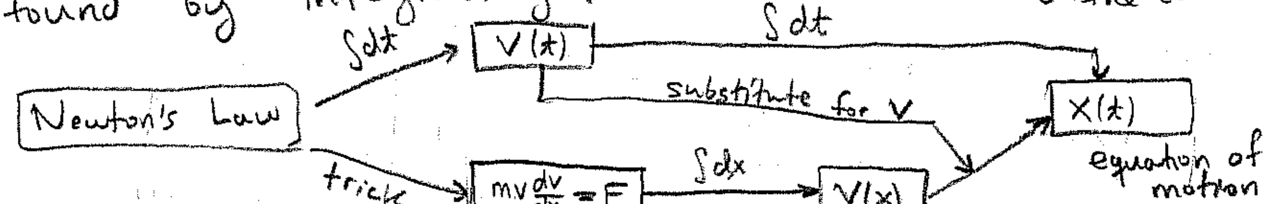
$$0 = \frac{-m}{10k} + C_2 \quad \therefore \quad C_2 = \frac{m}{10k}$$

Thus,

$$X = \frac{m}{k}\left(\frac{1}{10} - \frac{1}{V}\right)$$

$$= \boxed{\frac{m}{k}\left(\frac{1}{10} - \sqrt{2\left(\frac{1}{200} - \frac{kt}{m}\right)}\right) = X(t)}$$

You can check and see this is the same formula we found by integrating twice. Here's a diagram contrasting the calculations,



E36 The falling Raindrop: Imagine a drop falling through a cloud gathers water as it falls, let $m(t)$ be it's varying mass. Further assume as the drop gets bigger it gathers more & more mass proportionate to it's mass; $\frac{dm}{dt} = km$ for $k > 0$.

$F = ma$ is more generally $F = \frac{dP}{dt}$ when m varies.

$$F = \frac{dP}{dt} = \frac{d}{dt}(mv) = mg \quad \text{(fall's due to gravity)}$$

$$\frac{dm}{dt} v + m\frac{dv}{dt} = mg$$

$$kmv + m\frac{dv}{dt} = mg$$

$$\frac{dv}{dt} = \frac{mg - kmv}{m} = g - kv$$

$$\therefore \frac{dv}{kv - g} = -dt \Rightarrow \frac{dv}{v - g/k} = -k\,dt$$

Integrate both sides, $\ln|v - g/k| = -kt + \tilde{c}$ and exponentiate,

$$v - g/k = e^{-kt + \tilde{c}} = ce^{-kt}$$

$$\boxed{v(t) = ce^{-kt} + g/k}$$

The terminal velocity would be

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} (c\underbrace{e^{-kt}}_{\to 0} - g/k) = \boxed{g/k}$$

<u>Remark</u>: We took down as positive direction & Assumed no friction besides the water growth. Physically this amounts to under friction!

$$m\frac{dv}{dt} = ma = mg - kmv$$

When $v = g/k$ we have $mg - km\left(\frac{g}{k}\right) = 0$

terminal velocity happens when the <u>forces balance</u>