2^{nd} Order Linear Differential Eq^{n}'s with constant coefficients: §4.2–4.3

Now that we have studied 1^{st} order ODE's in some depth we'll study the simplest class of 2^{nd} order ODE's. Let us begin with the easiest case called "homogeneous"

$$\boxed{Ay'' + By' + Cy = 0} \quad (*)$$

What is a sol^{n}? Well let's make an ansatz, that is an educated guess; $y = e^{\lambda x}$ if this is the sol^{n} then note

$$y' = \lambda e^{\lambda x}$$
$$y'' = \lambda^2 e^{\lambda x}$$

Substituting yields:

$$A\lambda^2 e^{\lambda x} + B\lambda e^{\lambda x} + Ce^{\lambda} = (A\lambda^2 + B\lambda + C)e^{\lambda} = 0$$

Now since $e^{\lambda} \neq 0$ for any $\lambda \in \mathbb{C}$ we find the Characteristic Eq^{n} for $(*)$ namely

$$\boxed{A\lambda^2 + B\lambda + C = 0} \quad \underline{\text{Ch. Eq}^{n}}$$

Notice that we can <u>allways</u> solve this eq^{n} $(A \neq 0)$ with the quadratic formula:

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Where $B^2 - 4AC$ is the discriminant which discriminates which type of sol^{n} we get. Lets label the cases

I.) $B^2 - 4AC > 0$ distinct real roots

II.) $B^2 - 4AC = 0$ repeated real roots

III.) $B^2 - 4AC < 0$ Complex roots

With this in mind now we'll solve $(*)$ in each of the above cases. Notice since $(*)$ is 2^{nd} order we need <u>TWO</u> arbitrary $\int$ constants $\therefore$ <u>TWO</u> "linearly independent" sol^{n}'s, happily quad. eq^{n}'s give us <u>TWO</u> sol^{n}'s. We began looking for a sol^{n} and found the general sol^{n}! neat.

I.) $B^2 - 4AC > 0$: Distinct Real Roots

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \in \mathbb{R}$$

The sol^{n} is simply $\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$
since $\lambda_1, \lambda_2 \in \mathbb{R}$ these are just plain-old exponentials.

E37 $y'' + 3y' + 2y = 0 \Rightarrow \lambda^2 + 3\lambda + 2 = 0$

Thus $(\lambda+1)(\lambda+2) = 0 \therefore \underline{\lambda_1 = -1 \text{ and } \lambda_2 = -2}$ Sol^{n}'s to Ch. Eq^{n}.
(No need to use quad. eq^{n} if it factors, remember ?)

Hence $\boxed{y = c_1 e^{-x} + c_2 e^{-2x}}$ where c_1, c_2 are arbitrary real constants.

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II.) $B^2 - 4AC = 0$: Repeated Roots

$$\lambda_{1,2} = -B/2A \equiv \lambda$$

The sol^{n} is $\boxed{y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}}$. I'll not try to derive where the x comes from, rather we'll show it works, $y_2 = xe^{\lambda x}$

$$\frac{d}{dx}\left(xe^{\lambda x}\right) = e^{\lambda x} + \lambda x e^{\lambda x}$$

$$\frac{d^2}{dx^2}\left(xe^{\lambda x}\right) = \lambda e^{\lambda x} + \lambda e^{\lambda x} + x\lambda^2 e^{\lambda x}$$

$$\begin{aligned}
Ay_2'' + By_2' + Cy_2 &= A\left(2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}\right) + B\left(e^{\lambda x} + \lambda x e^{\lambda x}\right) + Cxe^{\lambda x} \\
&= A\left(\frac{-2B}{2A} e^{\lambda x} + \frac{B^2}{4A^2} x e^{\lambda x}\right) + B\left(e^{\lambda x} - \frac{B}{2A} x e^{\lambda x}\right) + Cxe^{\lambda x} \\
&= \cancel{-Be^{\lambda x}} + \cancel{\frac{B^2}{4A}} x e^{\lambda x} + \cancel{Be^{\lambda x}} - \cancel{\frac{B^2}{2A}} x e^{\lambda x} + \cancel{\frac{B^2}{4A}} x e^{\lambda x} \\
&= 0
\end{aligned}$$

Hence $y_1 = e^{\lambda x}$ satisfies (*) clearly, and $y_2 = xe^{\lambda x}$ satisfies (*) by above calculation. So also $y = c_1 y_1 + c_2 y_2$ will solve (*) since differentiation is a linear operation.

E38 $y'' - 2y' + y = 0$

$\Rightarrow \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0 \quad \therefore \underline{\underline{\lambda = 1}}$

$\therefore \quad \boxed{y = c_1 e^x + c_2 x e^x}$

III. $B^2 - 4AC < 0$: Complex Roots

Since it is customary to use explicitly real-valued fncts. we need to understand what $e^{\lambda x}$ means when $\lambda \in \mathbb{C}$.

$\boxed{\text{Th}^{\text{m}} \text{ (Euler's Id.) } e^{i\theta} = \cos\theta + i\sin\theta.}$

Additional Properties; these follow easily from Th$^{\text{m}}$, and ④ is a definition

① $e^{-i\theta} = \cos\theta - i\sin\theta$

② $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

③ $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

④ $e^{(a+ib)x} \equiv e^{ax}e^{ibx} = e^{ax}(\cos(bx) + i\sin(bx))$

With these facts in-hand lets derive the real-form of type III. sol$^{\text{n}}$s.
As in case I $\quad y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-B}{2A} \pm \frac{\sqrt{(-1)}\sqrt{4AC - B^2}}{2A}$$

$$= \frac{-B}{2A} \pm i\,\frac{\sqrt{4AC - B^2}}{2A}$$

$$= \alpha \pm i\beta$$

Where we defined $\alpha \equiv \frac{-B}{2A}$ & $\beta \equiv \frac{\sqrt{4AC - B^2}}{2A}$

Notice $\alpha, \beta \in \mathbb{R}$.

III. Continued

We are solving $Ay'' + By' + Cy = 0$ in the case $A\lambda^2 + B\lambda + C = 0$ has solution $\lambda = \alpha \pm i\beta$ for real constants α, β which are related to A, B, C by the formulas $\alpha = -B/2A$ and $\beta = \sqrt{4AC - B^2}/2A$.
We have shown that $y = e^{\lambda x}$ is a sol$^{\underline{n}}$ provided that λ is a solution to the characteristic eq$^{\underline{n}}$ $A\lambda^2 + B\lambda + C = 0$.
However, $y = e^{\lambda x} = e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$ is clearly a <u>complex - valued</u> <u>solution</u>. We wish to find <u>real - valued</u> <u>solutions</u>. (see Appendix on Complex Math for more on this)

FACT: A complex - valued solution y to a linear DEq$^{\underline{n}}$ contains two real-valued solutions, namely $y_1 = Re(y)$ and $y_2 = Im(y)$.

Thus, the general solution in case III is

$$y = c_1 Re(e^{\lambda x}) + c_2 Im(e^{\lambda x})$$

$$\boxed{y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x}$$

E39 Solve $y'' + 4y = 0$
Char. Eq$^{\underline{n}}$ is $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow \alpha = 0, \beta = 2$

$$\therefore \quad \boxed{y = c_1\cos(2x) + c_2\sin(2x)}$$

E40 Solve $y'' + 4y' + 5y = 0$
Char. Eq$^{\underline{n}}$ is $\lambda^2 + 4\lambda + 5 = 0 \Rightarrow (\lambda + 2)^2 + 1 = 0$

$$\Rightarrow \lambda = -2 \pm i$$

$$\Rightarrow \underline{\alpha = -2 \text{ and } \beta = 1}$$

$$\Rightarrow \boxed{y = c_1 e^{-2x}\cos(x) + c_2 e^{-2x}\sin(x)}$$

<u>Remark</u>: An alternate sol$^{\underline{n}}$ to **E40** is $y = b_1 e^{(-2+i)x} + b_2 e^{(-2-i)x}$.
The "trouble" with this sol$^{\underline{n}}$ is it is complex-valued. To fit initial data which is real we'll need to choose complex-valued b_1 & b_2.

Continuing on why we like to use $Re(e^{\lambda x})$, $Im(e^{\lambda x})$ rather than $e^{\lambda x}$ and $e^{\lambda^* x}$ as our solution set in the complex case: continuing to discuss $\boxed{E40}$

Suppose $y(0) = 0$ and $y'(0) = 1$. Let's try to use $y = b_1 e^{(-2+i)x} + b_2 e^{(-2-i)x}$:

$$y(0) = 0 \Rightarrow b_1 + b_2 = 0$$

$$y'(0) = 0 \Rightarrow \underbrace{(-2+i)b_1 + (-2-i)b_2}_{(-2+i)b_1 - (-2-i)b_1} = 1$$

$$(-2+i)b_1 - (-2-i)b_1 = 1$$

$$2ib_1 = 1 \quad \therefore \quad b_1 = \frac{1}{2i}$$

$$\Rightarrow b_2 = \frac{-1}{2i}$$

Thus the sol$^{\underline{n}}$ has the form $\boxed{y = \frac{1}{2i}e^{(-2+i)x} - \frac{1}{2i}e^{(-2-i)x}}$ — (*)

the boxed sol$^{\underline{n}}$ is a real-valued sol$^{\underline{n}}$, but perhaps this is not clear. Let me solve the same problem using the manifestly real general sol$^{\underline{n}}$: $y = c_1 e^{-2x}\cos(x) + c_2 e^{-2x}\sin(x)$

thus $y' = c_1\left(-2e^{-2x}\cos(x) - e^{-2x}\sin(x)\right) + c_2\left(-2e^{-2x}\sin(x) + e^{-2x}\cos(x)\right)$

$\underline{y(0) = 0}\,|\quad 0 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) \Rightarrow \underline{c_1 = 0}$.

$\underline{y'(0) = 1}\,|\quad 1 = c_2\left(-2e^0\sin(0) + e^0\cos(0)\right) \Rightarrow \underline{c_2 = 1}$.

Thus the sol$^{\underline{n}}$ is $\boxed{y = e^{-2x}\sin(x)}$ — (**)

(**) • $y = e^{-2x}\sin(x)$ is clearly real-valued.

(*) • $y = \frac{1}{2i}e^{(-2+i)x} - \frac{1}{2i}e^{(-2-i)x}$ is the same sol$^{\underline{n}}$ in disguise.

$$= e^{-2x}\left[\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\right]$$

$$= e^{-2x}\sin(x).$$

The <u>reason</u> we work with $y = c_1 \underbrace{e^{\alpha x}\cos\beta x}_{y_1} + c_2 \underbrace{e^{\alpha x}\sin\beta x}_{y_2}$ is that y_1 and y_2 are themselves real sol$^{\underline{n}}$'s and consequently c_1, c_2 will be real when chosen to fit <u>real</u> initial data.

More examples of homogeneous eq^{n}'s $Ay''+By'+Cy=0$ (§4.2-4.3)

E41 $2y''+6y'-11y=0$

$2\lambda^2+6\lambda-11=0 \Rightarrow \lambda = \frac{-6\pm\sqrt{36+4(2)(11)}}{4} = \frac{-6\pm\sqrt{124}}{4}$

Thus $\lambda = \frac{-3\pm\sqrt{31}}{2}$ which is distinct real case, (I.)

$$\boxed{y = c_1 e^{\frac{-3+\sqrt{31}}{2}x} + c_2 e^{\frac{-3-\sqrt{31}}{2}x}}$$

E42 $\frac{d^2r}{d\theta^2}+16r=0$

$\lambda^2+16=0 \Rightarrow \lambda = \pm 4i$ or $\alpha=0$ and $\beta=4$

$$\boxed{r(\theta) = c_1\cos(4\theta)+c_2\sin(4\theta)}$$

E43 $\varphi''+2\varphi'+5\varphi=0$ where $\varphi' \equiv \frac{d\varphi}{dt}$.

$\lambda^2+2\lambda+5=0$

$\lambda = \frac{-2\pm\sqrt{4-4(5)}}{2} = -1\pm\frac{\sqrt{-16}}{2} = -1\pm 2i$, $\alpha=-1$ & $\beta=2$

$$\boxed{\varphi(t) = e^{-t}\left(a\cos(2t)+b\sin(2t)\right)}$$

E44 $\psi''-16\psi=0$ where $\frac{d\psi}{dx} \equiv \psi'$,

$\lambda^2-16=0 \Rightarrow \lambda=\pm 4$

$$\boxed{\psi = c_1e^{4x}+c_2e^{-4x}}$$

Remark: There is a prettier way to phrase this case ($B=0$) using the hyperbolic trig fncts.

$\cosh(x) = \frac{1}{2}(e^x+e^{-x})$ & $\sinh(x) = \frac{1}{2}(e^x-e^{-x})$

$\psi(x) = A\cosh(4x)+B\sinh(4x)$ ← equivalent sol^{n} to

$= \frac{A}{2}(e^{4x}+e^{-4x})+\frac{B}{2}(e^{4x}-e^{-4x})$

$= \left(\frac{A}{2}+\frac{B}{2}\right)e^{4x}+\left(\frac{A}{2}-\frac{B}{2}\right)e^{-4x}$

$= c_1e^{4x}+c_2e^{-4x}$

$\boxed{E45}$. Given $y(0) = 3$ and $y'(0) = 0$, solve

$$y'' + 2y' - 8y = 0$$

$$\lambda^2 + 2\lambda - 8 = (\lambda+4)(\lambda-2) = 0 \quad \therefore \lambda_1 = -4, \ \lambda_2 = 2$$

$$y(x) = c_1 e^{-4x} + c_2 e^{2x}$$ (could use t just as well here,

$$y'(x) = -4c_1 e^{-4x} + 2c_2 e^{2x}$$

$$y(0) = c_1 + c_2 = 3 \longrightarrow -2c_1 - \cancel{2c_2} = -6$$

$$y'(0) = -4c_1 + 2c_2 = 0 \qquad -4c_1 + \cancel{2c_2} = 0$$

$$-6c_1 = -6$$

$$\therefore c_1 = 1$$

$$c_2 = 3 - c_1 = 3 - 1 = 2 = c_2$$

$$\therefore \boxed{y = e^{-4x} + 2e^{2x}}$$

Remark: there are a few problems in the homework in § 4.2, 4.3 on linear independence and springs. These problems are covered by TEST II. I will discuss linear independence after TEST I. Sufficient to say for now that $\{y_1, y_2\}$ we have found from cases I, II & III are linearly independent for all of $\mathbb{R}$. This continues to be true for the $n > 2$ cases we discuss next,

Higher Order Homogeneous Constant Coefficient ODEs : §6.2

I will systematically derive the results I use below in a later lecture. However, for now, our focus is not on the general theory. Here's a sketch of the logic here:

Given: $y^{(n)}(x) + P_{n-1}\, y^{(n-1)}(x) + \cdots + P_2 y'' + P_1 y' + P_0 y = 0$
for real constants $P_0, P_1, \ldots, P_{n-1}$.

FIND CHARACTERISTIC EQUATION: $\lambda^n + P_{n-1}\lambda^{n-1} + \cdots + P_2\lambda^2 + P_1\lambda + P_0 = 0$

Solve CHAR. EQ$^{\underline{n}}$: Get $\lambda_1, \lambda_2, \ldots, \lambda_n$ roots, some may be repeated and/or conjugate pairs

WRITE GENERAL SOLUTION: $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

I'll illustrate how to find $y_1, y_2, \ldots, y_n$ by example.

E46 Solve

$$y''' + 5y'' + 6y' = 0$$
$$\lambda^3 + 5\lambda^2 + 6\lambda = 0$$
$$\lambda(\lambda^2 + 5\lambda + 6) = 0$$
$$\lambda(\lambda+2)(\lambda+3) = 0 \Rightarrow \underline{\lambda_1 = 0,\ \lambda_2 = -2,\ \lambda_3 = -3}$$
$$\Rightarrow \boxed{y = c_1 + c_2 e^{-2x} + c_3 e^{-3x}}$$

E47

$$y'''' - y = 0$$
$$\lambda^4 - 1 = 0$$
$$(\lambda^2+1)(\lambda^2-1) = 0$$
$$\lambda = \pm i \text{ or } \lambda = \pm 1$$
$$\boxed{y = c_1\cos(x) + c_2\sin(x) + \underbrace{c_3 e^{x} + c_4 e^{-x}}}$$

could use $\bar{c}_3\cosh(x) + \bar{c}_4\sinh(x)$ if you prefer.

E48 $y'''' = 0$

$$\lambda^4 = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$$

$$\Rightarrow \boxed{y = c_1 + c_2 x + c_3 x^2 + c_4 x^3}$$

NOTATION: THE BIG $D = d/dx$. We can denote a differentiation such as $y'' = D^2[y]$ or $y' + y = (D+1)[y]$. Multiplication of $(D+2)(D+3)[y]$ means composition of the operations,

$$\begin{aligned}(D+2)(D+3)[y] &= (D+2)[y' + 3y] \\ &= D[y' + 3y] + 2(y' + 3y) \\ &= y'' + 3y' + 2y' + 6y \\ &= y'' + 5y' + 6y \\ &= (D^2 + 5D + 6)[y]\end{aligned}$$

This is just an example but it is true in general that if we write the differential eq$^{\underline{n}}$ as a polynomial in $D = d/dx$ then we can factor it as if it were an ordinary polynomial in λ. It turns out we can either do algebra in λ or do algebra in D. Again, I'll justify these comments in more depth later.

E49 $(D+1)^3[y] = 0$ same as $y''' + 3y'' + 3y + y = 0$

$$(\lambda+1)^3 = 0 \quad \therefore \quad \lambda = -1 \text{ three times.}$$

$$\Rightarrow \boxed{y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}}$$

E50 It is a gift when the given eq^{n} is already factored. This is part of the beauty of the D-notation, the prime notation makes such gifts harder to give,

$$(D+1)(D-2)(D^2+1)[y] = 0$$

$$(\lambda+1)(\lambda-2)(\lambda^2+1) = 0$$

$$\Rightarrow \lambda_1 = -1, \ \lambda_2 = 2, \ \lambda_{3,4} = \pm i$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos(x) + c_4 \sin(x)}$$

E51 Solve $D(D^2+4D+5)^2[y] = 0$

$$\Rightarrow D\left((D+2)^2+1\right)^2[y] = 0$$

$$\Rightarrow \lambda\left[(\lambda+2)^2+1\right]^2 = 0$$

pragmatically this step is redundant. However, I do not look kindly upon statements such as "$\underbrace{D = -2 \pm i}_{\text{nonsense!}}$"

$$\lambda_1 = 0, \quad \lambda_{2,4,5,6} = -2 \pm i$$

$$\boxed{\begin{aligned} y = c_1 &+ c_2 e^{-2x}\cos x + c_3 e^{-2x}\sin x \\ &+ c_4 x e^{-2x}\cos x + c_5 x e^{-2x}\sin x \end{aligned}}$$

notice we include an x because $\lambda = -2 \pm i$ is repeated.

E52 $(D^2+4)^3[y] = 0$

$$\Rightarrow \lambda = \pm 2i \text{ repeated three times}$$

$$\Rightarrow \boxed{y = c_1\cos 2x + c_2 \sin 2x + c_3 x\cos 2x + c_4 \sin 2x + c_5 x^2 \cos 2x + c_6 x^2 \sin 2x}$$