Linear Independence of Functions:

We wish to determine when we have found all possible sol^n's to a given DEq^n. Even in the 1^{st} order case there are certain pathological examples that do not allow for a unique sol^n. However, often we find that if we have an n^{th} order DEq^n and n-initial conditions then $\exists!$ solution in some neiborhood of the initial point. The sol^n will consist of n- distinct functions. What do I mean by this? We need a concept to distinguish sol^n's which are "genuinely" different,

Def^n/ A set of functions $f_1, f_2, \dots, f_n$ are linearly independent on a subset I of $\mathbb{R}$ iff dom(f_1), dom(f_2), .., dom(f_n) includes I and for $x \in I$,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \implies c_1 = c_2 = \cdots = c_n = 0.$$

When these conditions are met then $\{f_1, f_2, .., f_n\}$ is said to be a linearly independent (LI) set on I. Conversely if for $x \in I$,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \not\Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

then $\{f_1, f_2, \dots, f_n\}$ is said to be a linearly dependent set of functions on I and we can find constants $c_1, c_2, \dots, \widehat{c_k}, \dots, c_n$ not all zero such that (deleted from list)

$$f_k(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + \widehat{c_k f_k(x)} + \cdots + c_n f_n(x)$$

$\boxed{E53}$ Let $f_1(x) = e^x$ and $f_2(x) = xe^x$. Is $\{f_1, f_2\}$ LI on $\mathbb{R}$?

Suppose $c_1 e^x + c_2 x e^x = 0$ then $c_1 + c_2 x = 0$. Let $x = 0$ then we find $\underline{c_1 = 0}$. I follows from $x = 1$ that $c_1 + c_2 = 0$ but we already know $c_1 = 0 \therefore \underline{c_2 = 0}$.

$\boxed{E54}$ Let $f_1(x) = \sin(x)$ and $f_2(x) = 3\sin(x)$. Is $\{f_1, f_2\}$ LI on $\mathbb{R}$? Notice $3f_1(x) - f_2(x) = 3\sin(x) - 3\sin(x) = 0 \quad \forall x \in \mathbb{R}$, thus $\{f_1, f_2\}$ is linearly dependent set of functions on $\mathbb{R}$. Moreover we can write $f_2(x) = 3f_1(x) \quad \forall x \in \mathbb{R}$.

$\boxed{E55}$ Consider $f_1(x) = |x|$ and $f_2(x) = x$ where $dom(f_1) = dom(f_2) = \mathbb{R}$. Does $\{f_1, f_2\}$ form a linearly independent set of functions on $\mathbb{R}$? Suppose that for $x \in \mathbb{R}$,

$$c_1|x| + c_2 x = 0$$

Then $x = 1 \Rightarrow c_1 + c_2 = 0 \quad \therefore \quad c_2 = -c_1$. Thus $c_1|x| = c_1 x$.
Then $x = -1 \Rightarrow c_1 = -c_1 \Rightarrow 2c_1 = 0 \Rightarrow \underline{c_1 = 0 \Rightarrow c_2 = 0}$.

<u>Remark</u>: for two functions it's often convenient to think of linear dependence of $f(x)$ and $g(x)$ in terms of the following criteria:

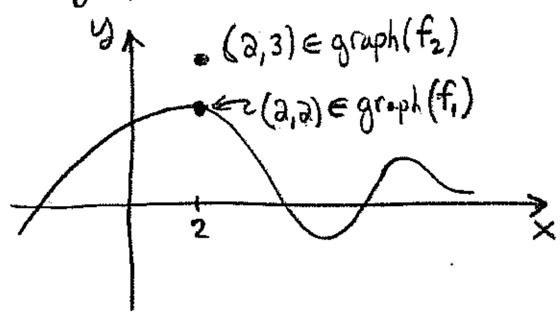
$\{f, g\}$ forms a linearly dependent set of functions on I $\iff$ There exists a constant $k \neq 0$ such that for all $x \in I$, $f(x) = k\, g(x)$

$\boxed{E56}$ Consider $f_1(x) = |x|$ and $f_2(x) = x$. Is $\{f_1, f_2\}$ a LI set of vectors on $(0,1)$? Observe that for $x > 0$,

$$f_1(x) = |x| = x = f_2(x)$$

Thus $f_1(x) = f_2(x) \quad \forall x \in (0,1)$. Therefore $\{f_1, f_2\}$ is <u>not</u> LI, instead it is a linearly dependent set of functions on $(0,1)$.

$\boxed{E57}$ The graphs of $f_1(x)$ and $f_2(x)$ are <u>identical</u> except at $x = 2$ where $f_1(2) = 2$ and $f_2(2) = 3$. If $2 \in I$ then $\{f_1, f_2\}$ are LI on I. However, if $2 \notin I$ then $\{f_1, f_2\}$ are linearly dependent on I.



Linear Independence of Functions vs. LI in $\mathbb{R}^n$

You may not have taken linear algebra by this time, if so the idea of linear indepence is probably new for you. Linear independence of two functions is really only defined with respect to some <u>interval</u>. In contrast, $\vec{V}_1, \vec{V}_2, \dots, \vec{V}_m \in \mathbb{R}^n$ are LI iff

$$c_1\vec{V}_1 + c_2\vec{V}_2 + \cdots + c_m\vec{V}_m = 0 \Rightarrow c_1 = c_2 = \cdots = c_m = 0.$$

There's nothing about any interval. The distinction is that <u>functions</u> are paired with their <u>domains</u> so we must keep that in mind when discussing them as <u>abstract vectors</u>. Let me contrast the vector space of all smooth functions on $\mathbb{R}$ (denoted $C^\infty(\mathbb{R})$) with n-dimensional space (denoted $\mathbb{R}^n$)

$\mathbb{R}^n$ is n-dimensional, it has <u>basis</u> of n-LI unit-vectors $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,0,\dots,0), \dots, e_n = (0,\dots,1)$
$C^\infty(\mathbb{R})$ is ∞-dimensional, it does not possess a LI finite set of functions which <u>span</u> all of $C^\infty(\mathbb{R})$. There are lots and lots of LI <u>sets of functions</u> inside $C^\infty(\mathbb{R})$ which will span some <u>subspace</u> of $C^\infty(\mathbb{R})$. Examples, $\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ LI set $\{e^x, xe^x, x^2e^x, x^3e^x\}$ LI set $\{1, x, x^2\}$ LI set, forms <u>basis</u> for <u>subspace</u> of up-to quadratic polynomials.

Linear independence of functions is a more abstract idea then the linear independence of vectors we <u>primarily</u> discuss throughout math 321. I mention this here because LI of functions is an <u>essential</u> idea for DEq$^{\underline{n}}$'s. Solutions to linear DEq$^{\underline{n}}$'s are only <u>distinct</u> if they are LI.

The Wronskian and Linear Independence:

Suppose that $f_1, f_2, \dots, f_n \in C^{n-1}(I) = $ $(n-1)$-differentiable fncts on I. Furthermore, suppose $f_1, f_2, \dots, f_n$ are LI on I. For each $x \in I$,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

It follows,

$$c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0$$
$$c_1 f_1''(x) + c_2 f_2''(x) + \cdots + c_n f''(x) = 0$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots$$
$$c_1 f_1^{(n-1)}(x) + c_2 f^{(n-1)}(x) + \cdots + c_n f^{(n-1)}(x) = 0$$

We find that for each $x \in I$

$$\left.\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & \cdots & f_n''(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} (*)$$

and $c_1 = c_2 = \cdots = c_n = 0$. Thus the system of equations has the <u>unique</u> sol$^{\underline{n}}$ $[c_1, c_2, \dots, c_n]^T = \vec{0}$. It follows from the theory of determinants (from linear algebra) that

$$W[f_1, f_2, \dots, f_n](x) \equiv \det \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \neq 0 \quad \text{for all } x \in I.$$

Th$^{\underline{m}}$/ Let $f_1, f_2, \dots, f_n \in C^{n-1}(I)$ then

$$\{f_1, f_2, \dots, f_n\} \text{ are LI on } I \Rightarrow W[f_1, f_2, \dots, f_n](x) \neq 0 \quad \forall x \in I$$

<u>Pf</u>: The discussion preceding Th$^{\underline{m}}$ proves $\Rightarrow$ direction.

<u>Remark</u>: $W \neq 0 \not\Rightarrow$ linear independence in general. However, if <u>all</u> the functions in W are sol$^{\underline{n}}$'s to a common DEq$^{\underline{n}}$ then the converse holds. Look ahead to 65 for a proof of the converse Th$^{\underline{m}}$ in the $n=2$ case. Next page we continue to discuss the subtleties of the Wronskian.

WRONSKIAN CONTINUED: (this page confronts a subtle issue)

The complementary Th$^{\underline{m}}$ for linear dependence fails for functions of class $C^{n-1}(I)$. For example,

$$f_1(x) = x^2 \quad , \quad f_2(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

these functions are clearly linearly independent, we cannot find k such that $f_2(x) = kf_1(x)$. Note

$$W[f_1, f_2](x) = \det\begin{bmatrix} x^2 & \mp x^2 \\ 2x & \mp 2x \end{bmatrix} = \mp(2x^3 - 2x^3) = 0.$$

where $\mp$ correspond to $x<0$ and $x \geq 0$ cases. We do have $f_2'(x) = \begin{cases} -2x & x<0 \\ 2x & x \geq 0 \end{cases}$ continuous, <u>however</u>

$f_2''(x) = \begin{cases} -2 & x<0 \\ 2 & x \geq 0 \end{cases}$ <u>not</u> continuous $\Rightarrow f_2 \notin C^2(\mathbb{R})$.

It would be nice if we could say

> Th$^{\underline{m}}$/ Let $f_1, f_2, \dots, f_n \in C^{n-1}(I)$ then $\{f_1, f_2, \dots, f_n\}$ are linearly dependent iff $W[f_1, \dots, f_n](x) = 0$ for all $x \in I$.

↑↑(THIS IS <u>FALSE</u> !)↑↑

Note the example preceding the Th$^{\underline{m}}$, I give two LI functions on $\mathbb{R}$ whose Wronskian is identically zero! There are numerous papers on this issue. Apparently Peano noted a counterexample like the one we gave back in the 1880's.

We can state a weaker Th$^{\underline{m}}$,

> Th$^{\underline{m}}$/ Let $f_1, f_2, \dots, f_n$ be <u>analytic</u> on I then $\{f_1, f_2, \dots, f_n\}$ are linearly dependent iff $W[f_1, \dots, f_n](x) = 0 \ \forall x \in I$.

In a nutshell, the stronger Th$^{\underline{m}}$ breaks down because a vanishing Wronskian $\not\Rightarrow$ the same k for all $x \in I$. $W[f_1, f_2](x) = 0 \ \forall x \in I$ could give $f_2(x) = k_1 f_1(x)$ for some x and $f_2(x) = k_2 f_1(x)$ for other x, such f_1, f_2 would be LI. The additional assumption of analyticity squashes this possibility.

Theory of Linear ODE's

These notes are based on chapters 4 & 6 of your text, rather than revisting the 2^{nd} order case alone, we treat it as a subcase of the n^{th} order ODE theory. I have notes on 2^{nd} order linear constant coefficient ODE's from Calculus II, those are given after I finish the general theory. There is some duplication since I reprove certain claims in the $n=2$ case later.

Def^n/ $Y^{(n)}(x) + P_1(x)Y^{(n-1)}(x) + \cdots + P_n(x)Y(x) = g(x)$ is a linear ordinary differential eq^n in standard form. $(*)$

- We usually assume $P_i(x)$ and $g(x)$ are continous on some interval $I \subseteq \mathbb{R}$.

- when $P_1(x), P_2(x), \dots, P_n(x)$ are all constants w.r.t. x this is said to be an n^{th} order <u>constant coefficient</u> ODE.

- when $g(x) \equiv 0$ this is a <u>homogeneous</u> ODE, otherwise if $g(x) \neq 0$ for some $x \in I$ (not necessarily all of I, just a point or two will do it) we say the DEq^n $(*)$ is <u>nonhomogeneous</u>

- When $g(x)$ is nontrivial we define the <u>auxillary eq^n</u> to be,

$$\boxed{Y^{(n)}(x) + P_1(x)Y^{(n-1)}(x) + \cdots + P_n(x)Y(x) = 0} \quad (**)$$

sol^n's to the auxillary eq^n play an important role in finding sol^n's to the DEq^n $(*)$. It is important to note that sol^n's to the auxillary eq^n are <u>not</u> sol^n's to $(*)$.

Th$^{\underline{m}}$/ Suppose that $P_1, P_2, \ldots, P_n, g$ are continous functions on (a, b) that contains x_0. Then the following IVP has a <u>unique</u> sol$^{\underline{n}}$ on (a, b).

$$\boxed{y^{(n)}(x) + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\, y(x) = g(x)} \quad (***)$$

$$\left.\begin{aligned} y(x_0) &= \gamma_0 \\ y'(x_0) &= \gamma_1 \\ &\vdots \\ y^{(n-1)}(x_0) &= \gamma_{n-1} \end{aligned}\right\} \text{ intial conditions}$$

This is the "<u>existence</u> and <u>uniqueness</u>" theorem. It's a little involved to prove, and it doesn't really tell us how to find the sol$^{\underline{n}}$ either. However, it's nice to know that a problem can be solved and we can use this Th$^{\underline{m}}$ to prove other important facts.

OPERATORS

An operator is a generalization of our notion of ordinary functions. An operator takes functions and maps them to new functions

"Usual" function

$$x \longrightarrow \boxed{f} \longrightarrow f(x) \qquad \begin{aligned} x &\in \mathbb{R} \\ f(x) &\in \mathbb{R} \end{aligned}$$

operator D

$$f \longrightarrow \boxed{D} \longrightarrow Df \qquad \begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ Df &: \mathbb{R} \to \mathbb{R} \end{aligned}$$

$$(Df)(x) \equiv \frac{d}{dx}(f(x))$$

Technically, D is also a function-valued function of functions, we call it an operator to draw attention to that fact. We can rewrite a DEq$^{\underline{n}}$ as an operator eq$^{\underline{n}}$, define

$$L \equiv D^n + P_1 D^{n-1} + \cdots + P_n \qquad \Big(\text{note } D^n = \underbrace{D \circ D \circ \cdots \circ D}_{\text{composition}}\Big)$$

Then $(*)$ is simply $L[y] = g$. It is customary to use [ ] to emphasize the input is a function and L is an operator which means $L[y]$ is a new function.

The operator L is a <u>linear operator</u>. Meaning

Def$^{\underline{n}}$ of Linear Operator

$$\boxed{\begin{array}{l} 1.)\ L[Y_1 + Y_2 + \cdots + Y_m] = L[Y_1] + L[Y_2] + \cdots + L[Y_m] \\ 2.)\ L[cY] = c\,L[Y] \quad \text{for any constant } c. \end{array}}$$

<u>Remark</u>: calculus is full of linear operations, definite integration, differentiation, Laplace Transform, ...

Now suppose we have a <u>homogeneous</u> linear $n^{\underline{th}}$ ODE in standard form on some interval I. We can compactly write it as $L[Y] = 0$ or putting in the x-dependence $L[Y](x) = 0$. Further suppose we have 2 sol$^{\underline{n}}$'s Y_1 and Y_2 notice,

$$\begin{aligned} L[C_1 Y_1 + C_2 Y_2] &= L[C_1 Y_1] + L[C_2 Y_2] && \text{used 1.} \\ &= C_1\, L[Y_1] + C_2\, L[Y_2] && \text{used 2.} \\ &= C_1\,(0) + C_2\,(0) && \text{using } Y_1 \,\&\, Y_2 \text{ are sol}^{\underline{n}}\text{'s} \\ &= 0. \end{aligned}$$

This proves $C_1 Y_1 + C_2 Y_2$ is also a sol$^{\underline{n}}$. So then we might wonder how many "independent" sol$^{\underline{n}}$'s are there for an $n^{\underline{th}}$ order homog. linear ODE?

> **PROPOSITION**: Every sol$^{\underline{n}}$ to $L[Y](x) = 0$ has the form $Y = C_1 Y_1 + C_2 Y_2 + \cdots + C_n Y_n$, where $L[Y_i](x) = 0$ $i = 1, 2, \ldots, n$ and $\{y_1, y_2, \ldots, y_n\}$ form a linearly independent set on $\mathbb{R}$.

Recall we have already defined linear independence in a previous section of the notes. We found that LI $\Rightarrow$ non-vanishing Wronskian. This will be important in the proof that follows

<u>Proof of Prop.</u>: Study $[D^n + P_1 D^{n-1} + \cdots + P_n][Y] = 0$. Suppose that we have an arbitrary sol$^{\underline{n}}$ $\phi(x)$ on (a,b). We seek to show $\phi(x) = c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n$ for appropriate choice of $c_1, c_2, \ldots, c_n$ (constants). Let $x_o \in (a,b)$ and consider that if we can satisfy the system of eq$^{\underline{n}}$'s,

$$\left.\begin{array}{l} c_1 Y_1(x_o) + c_2 Y_2(x_o) + \cdots + c_n Y_n(x_o) = \phi(x_o) \\ c_1 Y_1'(x_o) + c_2 Y_2'(x_o) + \cdots + c_n Y_n'(x_o) = \phi'(x_o) \\ \vdots \\ c_1 Y^{(n-1)}(x_o) + c_2 Y_2^{(n-1)}(x_o) + \cdots + c_n Y_n^{(n-1)}(x_o) = \phi^{(n-1)}(x_o) \end{array}\right\} (**)$$

then by the existence & <u>uniqueness</u> Th$^{\underline{m}}$ we'll be able to conclude that $\phi(x) = c_1 Y_1 + \cdots + c_n Y_n$ as they both solve the same IVP. We can rewrite $(**)$ in matrix form

$$Ac \equiv \underbrace{\begin{bmatrix} Y_1(x_o) & Y_2(x_o) & \cdots & Y_n(x_o) \\ \vdots & & & \vdots \\ Y_1^{(n-1)}(x_o) & Y_2^{(n-1)}(x_o) & \cdots & Y_n^{(n-1)}(x_o) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} \phi(x_o) \\ \vdots \\ \phi^{(n-1)}(x_o) \end{bmatrix}}_{b} \equiv b$$

This is a system of n-equations with n-unknowns $c_1, c_2, \ldots, c_n$. It has a unique sol$^{\underline{n}}$ (no matter what $\phi(x_o), \ldots, \phi^{(n-1)}(x_o)$ might be) provided the matrix A has $\det(A) \neq 0$.

$$\det(A) = \begin{vmatrix} Y_1(x_o) & Y_2(x_o) & \cdots & Y_n(x_o) \\ \vdots & & & \vdots \\ Y_1^{(n-1)}(x_o) & Y_2^{(n-1)}(x_o) & \cdots & Y_n^{(n-1)}(x_o) \end{vmatrix} \neq 0$$

(we're using a well known fact from linear algebra here)

So we can conclude that if $Y_1, Y_2, \ldots, Y_n$ satisfy this condition they will construct the general sol$^{\underline{n}}$ to $L[Y] = 0$.

Def$^{\underline{n}}$/ $f_1, f_2, \ldots, f_n$ be n-functions which are at least $(n-1)$-times differentiable. Then

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

<u>Wronskian</u> of $f_1, f_2, \ldots, f_n$

- as we discussed before, but now we see the connection to DEq$^{\underline{n}}$'s. The non-vanishing of the Wronskian gaurantees we have enough sol$^{\underline{n}}$'s to form the general sol$^{\underline{n}}$.

 WRONSKIAN (BABY-VERSION, see chapter 6 for more)

a.) $\begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} \overset{\text{def}^n}{\equiv} Y_1 Y_2' - Y_1' Y_2 \overset{\text{def}^n}{\equiv} W[Y_1, Y_2]$ (suppressing t-dependence)

def^n of determinant of 2×2 matrix

b.) Suppose $aY'' + bY' + cY = 0$ has sol^n's $Y_1(t)$ and $Y_2(t)$

- Prove that Y_1 & Y_2 are L.I. on an interval I if and only if $W[Y_1, Y_2](t) \neq 0$ $\forall t \in I$. Consider,

Suppose, $Y_1(t)\, Y_2'(t) - Y_1'(t) Y_2(t) = W[Y_1, Y_2](t) = f(t) \neq 0$ $\forall t \in I$.

Now assume $Y_2(t) = c Y_1(t)$ towards a contradiction. Notice

$$Y_2'(t) = c\, Y_1'(t)$$

Thus $W[Y_1, Y_2](t) = \cancel{Y_1(t)\, c\, Y_1'(t) - Y_1'(t)\, c\, Y_1(t)} = 0$ but this contradicts that $W[Y_1, Y_2](t) \neq 0$.

Next, suppose that Y_1 and Y_2 are L.I. Then suppose towards a contradiction that $W[Y_1, Y_2](t) = 0$ then we have,

$$Y_1(t)\, Y_2'(t) - Y_1'(t)\, Y_2(t) = 0$$

Now since Y_1 & Y_2 are LI it follows that they cannot be the zero functions, that means we'll can find some small nbhd where $Y_1(t) \neq 0$ $\forall t \in J$. (so we can ÷ by Y_1 on J)

Consider then that

$$\frac{d}{dt}\left[\frac{Y_2}{Y_1}\right] = \frac{Y_2' Y_1 - Y_2 Y_1'}{Y_1^2} = \frac{0}{Y_1^2} = 0$$

(holds for all t on J where ÷ by $Y_1(t)$ makes sense)

Thus on J we find $\frac{Y_2}{Y_1} = c \Rightarrow Y_2 = c Y_1$.

But we need $Y_2(t) = c Y_1(t)$ $\forall t \in I$ (J is contained in I).

We can extend to I because if $Y_2 = c Y_1$ $\forall t \in J \Rightarrow$ they share same intial conditions, then we know sol^n's to $aY'' + bY' + cY = 0$ are uniquely determined by particular intial conditions. Hence $Y_2 = c Y_1$ on I therefore Y_2 is <u>not</u> Linearly independent of Y_1, a contradiction //

assuming Y_1 & Y_2 are sol^n's to some 2nd order ODE

$\boxed{\text{Thus } W[Y_1, Y_2](t) \neq 0 \ \forall t \in I \iff Y_1 \,\&\, Y_2 \text{ L.I. on } I}$

Th^{m}/ Let $y_1, y_2, \dots, y_n$ be n-sol^n's on (a,b) of the Homogeneous DEq^n

$$\boxed{y^{(n)}(x) + P_1(x)\, y^{(n-1)}(x) + \cdots + P_{n-1}(x)\, y'(x) + P_n(x)\, y(x) = 0} \quad (**)$$

where $P_1, P_2, \dots, P_n$ are continuous on (a,b). If for some $x_0 \in (a,b)$ the Wronskian $W[y_1, y_2, \dots, y_n](x_0) \neq 0$ then every sol^n of $(**)$ has the form $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$.

In other words, if we can somehow find n-LI sol^n's to an n^{th} order linear ODE then the general sol^n always can be formed by a linear combination of the n-LI sol^n's.

Def^n/ A set of n-LI sol^n's to an n^{th} order DEq^n of the form $(*)$ called a fundamental solution set.

Reminder: I already showed you how to find the fund. sol^n set for constant coefficient linear ODE's, we've yet to justify those calculations completely, but at least now you know that we weren't missing any piece of the general sol^n. We had an n^{th} order polynomial and the n-roots gave us n-sol^n's. Technically we should check those sol^n's are linearly independent.

Th^{m}/ (NONHOMOGENEOUS LINEAR ODE'S). The general sol^n to

$$\boxed{y^{(n)}(x) + P_1(x)\, y^{(n-1)}(x) + \cdots + P_{n-1}(x)\, y'(x) + P_n(x)\, y(x) = g(x)} \quad (*)$$

has the form $y = y_h + y_p$ where y_h is a sol^n to $(**)$ and y_p is called the "particular sol^n" of $(*)$. Sometimes $(**)$ is called the "auxillarly" or "complementary" homogeneous eq^n to $(*)$.

Observations:

① we already know how to find y_h in the constant coefficient case.

② we need some methods to find y_p. Turns out we'll use two methods
- undetermined coefficients
- variation of parameters

③ Series techniques will allow us to treat non-constant coeff. cases.