GENERAL SOLUTION OF THE n^{th} ORDER CONSTANT COEFFICIENT ODEq^n

Earlier Pages OF MY NOTES EXPLORE THIS QUESTION AND MORE, THE PURPOSE OF THESE NOTES IS TO EMPHASIZE THE LOGIC WITHIN THOSE NOTES, HERE I'LL SIMPLY COLLECT THE MAIN IDEAS AND ASSEMBLE THEM TO DERIVE THE SOL^n TO THE n^{th} ORDER LINEAR CONST. COEFF. ODE.

GOAL: Given that $a_0, a_1, a_2, \dots, a_n$ are real constants find general sol^n of

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \qquad Eq^n \text{ (I)}$$

E58 $y'' + 5y' + 6y = 0$ {Many of you did these in calc II}

$\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+3)(\lambda+2) = 0 \quad \therefore \lambda_1 = -3, \ \lambda_2 = -2$

therefore (by things we'll derive today)

$$\boxed{y = c_1 e^{-3x} + c_2 e^{-2x}}$$

OBSERVATION (1): Eq^n (I) can be rewritten as an operator eq^n; $\boxed{L[y] = 0}$ where $L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_2 D^2 + a_1 D + a_0$ with $D \equiv d/dx$

E59: $y'' + 5y' + 6y = 0 \iff (D^2 + 5D + 6)y = 0$

$\iff L = D^2 + 5D + 6$ and $L[y] = 0$.

CLAIM (1): THE L FROM Eq^n (I) CAN BE FACTORED INTO n-factors $L = L_1 \circ L_2 \circ \cdots \circ L_n$ so $L[y] = L_1(L_2(\cdots(L_n(y))\cdots) = 0$.

E60 $L = D^2 + 5D + 6 = (D+3)(D+2) = L_1 L_2$

OBSERVATION ②: If $L_k[y] = 0$ for some particular k with $1 \le k \le n$ then y is also a sol$^{\underline{n}}$ to $L[y] = L_1 L_2 \cdots L_n[y] = 0$.

$\boxed{E61}$: $y_1 = e^{-3x}$ solves $L_1[y] = (D+3)[y] = 0$ since $(D+3)[e^{-3x}] = -3e^{-3x} + 3e^{-3x} = 0$. Notice that $L[e^{-3x}] = (D+2)(D+3)e^{-3x} = (D+2)(0) = 0$.

Remark: OBSERVATION ② says that we can break-up the $n^{\underline{th}}$ order ODE into n-parts. If we can solve $L_k[y] = 0$ for $k = 1, 2, \dots, n$ then we'll get n-sol$^{\underline{n}}$'s to Eq$^{\underline{n}}$ ①.

CLAIM ②: ACTUALLY A FEW CLAIMS,

(i.) A linear combination of sol$^{\underline{n}}$'s to $L[y] = 0$ is a sol$^{\underline{n}}$.

$$L[y_k] = 0 \quad k = 1, 2, 3, \dots, n \Rightarrow L[c_1 y_1 + c_2 y_2 + \cdots + c_n y_n] = 0$$

where $c_1, c_2, \dots, c_n$ are constants.

(ii.) The general sol$^{\underline{n}}$ to $L[y] = 0$ has the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where $y_1, y_2, \dots, y_n$ are n- "linearly independent" functions.

$\boxed{E62}$: $L = (D+3)(D+2) = L_1 L_2$ has $L_1[e^{-3x}] = 0$ and $L_2[e^{-2x}] = 0$ So $L[e^{-3x}] = 0$ and $L[e^{-2x}] = 0 \Rightarrow y = c_1 e^{-3x} + c_2 e^{-2x}$ solves $L[y] = 0$. Moreover, since $y_1 = e^{-3x}$ and $y_2 = e^{-2x}$ are linearly independent we have the general sol$^{\underline{n}}$.

Remark: linear independence is an important concept which I have discussed in depth in earlier notes. If you forget the earlier discussion simply take it to mean that the functions are distinct (they have graphs which have different shapes).

OBSERVATION 3: The L from Eq$^{\underline{n}}$ I may be factored into the form
$$L = a_n(D-\lambda_n)(D-\lambda_{n-1})\cdots(D-\lambda_3)(D-\lambda_2)(D-\lambda_1)$$
where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n-constants. These constants may be real or complex and possibly $\lambda_1 = \lambda_2$ etc.... If there are complex roots then they come in conjugate pairs. (if $\lambda_1 = 3+i$ then $\lambda_2 = 3-i$ for example)

CLAIM 3: for any $\lambda \in \mathbb{C}$, $(D-\lambda)[y]=0$ has the sol$^{\underline{n}}$. $y = e^{\lambda x}$

Proof: $$(D-\lambda)[e^{\lambda x}] = \frac{d}{dx}(e^{\lambda x}) - \lambda e^{\lambda x} = \lambda e^{\lambda x} - \lambda e^{\lambda x} = 0. //$$

: I explain the details of when $\lambda \in \mathbb{C}$ in Complex Appendix. Short story, in $\mathbb{C}$ calculus works the same for functions $f : \mathbb{R} \to \mathbb{C}$.

CLAIM 4: $(D-\lambda)^2[y]=0$ has the sol$^{\underline{n}}$ $y_2 = xe^{\lambda x}$ and $y_1 = e^{\lambda x}$

Proof: $$\begin{aligned}(D-\lambda)^2[xe^{\lambda x}] &= (D-\lambda)\left[\tfrac{d}{dx}(xe^{\lambda x}) - \lambda x e^{\lambda x}\right] \\ &= (D-\lambda)[e^{\lambda x} + x\lambda e^{\lambda x} - \lambda x e^{\lambda x}] \\ &= (D-\lambda)[e^{\lambda x}] \\ &= 0. //\end{aligned}$$

CLAIM 5: $(D-\lambda)^m[y]=0$ has m-sol$^{\underline{n}}$'s
$y_m = x^{m-1}e^{\lambda x}$, $y_{m-1} = x^{m-2}e^{\lambda x}, \ldots, y_2 = xe^{\lambda x}, y_1 = e^{\lambda x}$

Remark: these sol$^{\underline{n}}$'s $y_m, y_{m-1}, \ldots, y_2, y_1$ are linearly independent. Sometimes they are complex sol$^{\underline{n}}$'s, we wish to find real sol$^{\underline{n}}$'s. This requires a little discussion

CLAIM 6 If y is a complex-valued sol^n to $L[y]=0$ where $y = Re(y) + i\,Im(y)$ then both $y_1 = Re(y)$ and $y_2 = Im(y)$ are real-valued sol^n's to $L[y]=0$. Hence, every complex sol^n has two (linearly independent) real sol^n's.

$\boxed{E63}$: $y'' + y = 0 \iff (D^2+1)[y] = 0$
$\iff (D-i)(D+i)[y] = 0$

now $(D-i)[y]$ has sol^n $y = e^{ix}$. By Euler's Identity we have that $e^{ix} = \cos(x) + i\sin(x)$. Thus

$$y = \cos(x) + i\sin(x)$$

we can read off that $Re(y) = \cos(x)$ & $Im(y) = \sin(x)$. These real sol^n's give us the general sol^n $y = c_1\cos(x) + c_2\sin(x)$.

What about $(D+i)$ you ask? Well that factor gives the same sol^n's since $(D+i)[y]=0 \Rightarrow y = e^{-ix}$ then $y = e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$ and we can read-off $Re(y) = \cos(x)$ and $Im(y) = -\sin(x)$. These are the "same" (upto linear independence) functions we already found from $(D-i)$.

Digression to answer common question.

PROPERTIES OF COMPLEX EXPONENTIAL

If $\lambda = \alpha + i\beta$ then we find

$$e^{\lambda x} = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$

then $e^{\lambda x} = e^{\alpha x}\cos\beta x + i e^{\alpha x}\sin\beta x$ and we can see

$$\boxed{Re(e^{\lambda x}) = e^{\alpha x}\cos\beta x \quad\Big|\quad Im(e^{\lambda x}) = e^{\alpha x}\sin\beta x}$$

(see the Complex Appendix Notes for more details on Euler's Identity)

<u>Remark</u>: <u>CLAIM 6</u> for $\lambda = \alpha + i\beta$ (here α, β are assumed to be real) we find <u>two</u> real sol$^{\underline{n}}$'s hidden inside $y = e^{\lambda x}$ namely

$$y_1 = \mathrm{Re}\, y = e^{\alpha x}\cos\beta x$$
$$y_2 = \mathrm{Im}\, y = e^{\alpha x}\sin\beta x$$

the general sol$^{\underline{n}}$ to $(D-\lambda)(D-\lambda^*)[y]=0$ is $y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$.

(Here $\lambda^* = \alpha - i\beta$, the factor $(D-\lambda^*)$ has same sol$^{\underline{n}}$'s as $D-\lambda$ so it is sufficient to find the sol$^{\underline{n}}$'s to $D-\lambda$.)

again the digression

<u>CLAIM 7</u> If we have a complex sol$^{\underline{n}}$ $y = x^m e^{\lambda x}$ then there are two real sol$^{\underline{n}}$'s hidden inside y, namely $\mathrm{Re}\, y$ and $\mathrm{Im}\, y$ which we can easily calculate

$$y_1 = \mathrm{Re}\, y = x^m e^{\alpha x}\cos\beta x$$
$$y_2 = \mathrm{Im}\, y = x^m e^{\alpha x}\sin\beta x$$

where again we have assumed $\lambda = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$.

<u>Concluding Thoughts</u>: we see Eq$^{\underline{n}}$ (I) can be rewritten in terms of a polynomial in $D = d/dx$ that is $L = P(D)$ where $P(D) = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0$. Then we <u>factor</u> P which of course reflects the zeroes of $P(D)$. For different types of factors we get either e^{ax}, $x^m e^{ax}$, $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$ or $x^m e^{\alpha x}\cos\beta x$ or $x^m e^{\alpha x}\sin\beta x$. There will be n- of these functions which we can then use to form the general sol$^{\underline{n}}$.

<u>Observation</u>: We can factor $P(\lambda)$ instead of $P(D)$, get same roots!

$$\begin{aligned} P(e^{\lambda x}) &= a_n D^n e^{\lambda x} + \cdots + a_2 D^2 e^{\lambda x} + a_1 D e^{\lambda x} + a_0 e^{\lambda x} \\ &= a_n \lambda^n e^{\lambda x} + \cdots + a_2 \lambda^2 e^{\lambda x} + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} \\ &= (a_n \lambda^n + \cdots + a_2\lambda^2 + a_1\lambda + a_0)\, e^{\lambda x} \\ &= P(\lambda) e^{\lambda x} \quad \therefore \quad \boxed{P(e^{\lambda x}) = 0 \iff P(\lambda) = 0} \end{aligned}$$

Solving n^{th} order constant coefficient linear ODEs

We wish to solve the following DEq^{n}:

$$\boxed{y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0} \quad \text{Eq}^{n}(1)$$

Here we assume $a_1, \ldots, a_{n-1}, a_n$ are real constants which certainly are continous on all of $\mathbb{R}$ so once we have n-intial conditions we know by existence/uniqueness Th^{m} the sol^{n} will exist and be defined for all $\mathbb{R}$. So let's find it, recast Eq^{n}(1) as an operator eq^{n}

$$\boxed{\left(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n\right) y \equiv L[y] = 0} \quad \text{Eq}^{n}(2)$$

Notice that the linear operator L is a polynomial in the operator $D \equiv \frac{d}{dx}$.

$$\boxed{L = P(D) = D^n + a_1 D^{n-1} + \cdots + a_n}$$

This polynomial P has real coefficients $a_1, \ldots, a_n$ so we can factor it as discussed in ALGEBRA REFRESHER

$$0 = L[y] = (D-\lambda_1)^{m_1} \cdots (D-\lambda_r)^{m_r} (D^2 + B_1 D + C_1)^{n_1} \cdots (D^2 + B_s D + C_s)^{n_s} y$$

Solving this eq^{n} is easy with what we know. Notice that all we need for y to be a sol^{n} is that one of the factors annihilate it. In total we get n-LI sol^{n}'s from the various factors

$\boxed{\text{linear factors}}$ $(D-\lambda_1)^{m_1} \Rightarrow y_1 = e^{\lambda_1 x}, \ldots, y_{m_1} = x^{m_1 - 1} e^{\lambda_1 x}$

$\boxed{\text{irreducible quad. factors}}$ $(D^2 - B_1 D + C_1)^{n_1} = (D - (\alpha + i\beta))^{n_1} (D - (\alpha - i\beta))^{n_1}$

$$\Rightarrow y_{M+1} = e^{\alpha x}\cos\beta_1 x, \quad y_{M+2} = e^{\alpha x}\sin\beta_1 x$$
$$y_{M+3} = x e^{\alpha_2 x}\cos\beta_2 x, \quad y_{M+4} = x e^{\alpha_2 x}\sin\beta_2 x, \ldots$$

Where I made up $M \equiv m_1 + m_2 + \cdots + m_r$ to keep the labeling correct. So we find in total n-LI sol^{n}'s to Eq^{n}(1), we'll write the general sol^{n} in all it's glory on the next page

Let us write the general sol^{n} to Eq^{n} (1),

$$\boxed{\begin{aligned} y = {} & c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} + \cdots + c_{m_1} x^{m_1 - 1} e^{\lambda_1 x} + \cdots \\ & + c_{M - m_r} e^{\lambda_{m_r} x} + \cdots + c_M x^{m_r - 1} e^{\lambda_{m_r} x} + \\ & + c_{M+1} e^{\alpha_1 x} \cos \beta_1 x + c_{M+2} e^{\alpha_1 x} \sin \beta_1 x + \cdots \\ & + c_{M + 2n_1 - 1} x^{n_1 - 1} e^{\alpha_1 x} \cos \beta_1 x + c_{M + 2n_1} x^{n_1 - 1} e^{\alpha_1 x} \sin \beta_1 x + \cdots \\ & + c_{n - 2n_s} e^{\alpha_s x} \cos \beta_s x + c_{n - 2n_s + 1} e^{\alpha_s x} \sin \beta_s x + \cdots \\ & + c_{n-1} x^{n_s - 1} e^{\alpha_s x} \cos \beta_s x + c_n x^{n_s - 1} e^{\alpha_s x} \sin \beta_s x \end{aligned}}$$

<u>Remark</u>: I don't remember this formula, instead I use the principles that led us to it to assemble sol^{n}'s for particular examples.

<u>Remark</u>: there are ∞ly many possible sol^{n}'s until we are supplied the needed n- initial conditions.

<u>Auxillary Eq^{n}</u>:

We have used operator arguments to assemble this sol^{n} but we could just as well supposed that $e^{\lambda x}$ was a sol^{n} to find

$$L[e^{\lambda x}] = P(\lambda) e^{\lambda x}$$

Where this is the same polynomial we found in D. Clearly $L[e^{\lambda x}] = 0$ iff $P(\lambda) = 0$. In practice I prefer to factor the auxillary eq^{n} in terms of λ as opposed to a polynomial in D.
- (The text uses "r" instead of "λ" so be warned.) •

- I'll factor $P(\lambda)$ to find $\lambda_1, \lambda_2, \ldots, \lambda_r$ and $\alpha_1, \beta_1, \ldots, \alpha_s, \beta_s$ and the multiplicities.

Examples (Now Justified!)

E64 $y'''(x) = 0$

$\lambda^3 = 0 \Rightarrow \boxed{y = c_1 + c_2 x + c_2 x^2}$ (since $e^{0 \cdot x} = 1$)

E65 $y'' + y = 0$

$\lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i = \alpha \pm i\beta$

identify $\alpha = 0$ & $\beta = 1$ $\therefore$ $\boxed{y = c_1 \cos(x) + c_2 \sin(x)}$

E66 $y''' + 3y'' + 3y' + 3y = 0$

$\lambda^3 + 3\lambda^2 + 3\lambda + 3 = 0$

$(\lambda + 1)^3 = 0$

$\lambda = -1$ with multiplicity 3. $\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}}$

E67 $y'''' + 2y'' + y = 0$

$\lambda^4 + 2\lambda^2 + 1 = 0$

$(\lambda^2 + 1)^2 = 0$

$\lambda = \pm i$ with multiplicity 2.

$\therefore$ $\boxed{y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x}$

E68 $y''' + 5y'' + 6y' = 0$

$\lambda^3 + 5\lambda^2 + 6\lambda = 0$

$\lambda(\lambda^2 + 5\lambda + 6) = \lambda(\lambda + 3)(\lambda + 2) = 0$

$\Rightarrow \lambda_1 = 0, \ \lambda_2 = -3, \ \lambda_3 = -2$

$\therefore$ $\boxed{y = c_1 + c_2 e^{-3x} + c_3 e^{-2x}}$

E69 $y' = y$

$\lambda = 1$ $\therefore$ $\boxed{y = c_1 e^{x}}$

E70 $y'' - y = 0$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow y = c_1 e^{x} + c_2 e^{-x}$

<u>Remark:</u> Consider what we've done, we've changed the problem of unraveling a complicated differential eq^{n} to the relatively easy problem of factoring a polynomial. Algebra has replaced calculus here, it's really quite amazing that constant coefficient linear ODEs are easy