Non-Homogeneous 2nd order Linear ODEq^{n}'s:

We have completely solved $aY'' + bY'' + cY = 0$ by solving the algebra problem $a\lambda^2 + b\lambda + c = 0$ to get $\lambda_1, \lambda_2 \Rightarrow Y = c_1 Y_1 + c_2 Y_2$ where Y_1 & Y_2 must be one of the following,

$$\boxed{\begin{array}{l} \text{I.)}\ e^{\lambda_1 x},\ e^{\lambda_2 x} \\ \text{II.)}\ e^{\lambda x},\ xe^{\lambda x} \\ \text{III.)}\ e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x \quad (\lambda_{1,2} = \alpha \pm i\beta) \end{array}}$$

these are the "fundamental sol^{n}'s"

Now we will see how to solve a certain class of eq^{n}'s with a fairly simple non-homogeneous (or forcing term)

$$\boxed{aY'' + bY' + cY = g(x)}$$

The general sol^{n} will have the form:

$$Y = Y_h + Y_p$$

Homogeneous Sol^{n}

$(aY_h'' + bY_h' + cY_h = 0)$

$Y_h = c_1 Y_1 + c_2 Y_2$

{We already learned this in the last section}

Particular Sol^{n}

$(aY_p'' + bY_p' + cY_p = g(x))$

{We'll find Y_p by the method of undetermined coefficient's or later, variation of parameters and/or series techniques}

I'll show a few examples then we'll discuss the annihilator method which justifies the method of undetermined coefficients

E71 $Y'' + 9Y = \sin(x)$ find general sol^{n}:

$\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i \therefore Y_h = c_1 \sin(3x) + c_2 \cos(3x)$

Next guess

$$Y_p = A\sin(x) + B\cos(x)$$
$$Y_p' = A\cos(x) - B\sin(x)$$
$$Y_p'' = -A\sin(x) - B\cos(x) = -Y_p$$

Remark: the "undetermined" coefficients are A and B

Substitute into our original eq^{n}:

$$Y_p'' + 9Y_p = -Y_p + 9Y_p = 8A\sin(x) + 8B\cos(x) = \sin(x)$$

Thus comparing coeff's $\Rightarrow A = 1/8$ & $B = 0$

$$\boxed{Y = c_1 \sin(3X) + c_2 \cos(3X) + \frac{1}{8}\sin(x)}$$

E 72

(2.) $y'' + 5y' + 6y = 12e^x + 6x + 11$

to begin find y_c.

$\lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0 \quad \therefore \lambda_1 = -3, \; \lambda_2 = -2$

$\Rightarrow \; y_c = c_1 e^{-3x} + c_2 e^{-2x}$

now we find the particular sol^{n} using the method of undetermined coefficients. Begin with the educated guess

$$y_p = Ae^x + Bx + C$$
$$y_p' = Ae^x + B$$
$$y_p'' = Ae^x$$

$$y_p'' + 5y_p' + 6y_p = 12e^x + 6x + 11$$
$$Ae^x + 5(Ae^x + B) + 6(Ae^x + Bx + C) = 12e^x + 6x + 11$$
$$e^x(A + 5A + 6A) + x(6B) + 5B + 6C = e^x(12) + x(6) + 11$$

Equate Coefficients of e^x, x and constants,

$e^x: \; 12A = 12 \Rightarrow A = 1$

$x: \; 6B = 6 \Rightarrow B = 1$

$x^0: \; 5B + 6C = 11 \Rightarrow 6C = 11 - 5 = 6 \Rightarrow C = 1$

So we find the general sol^{n} $y_g = y_c + y_p$ is

$$\boxed{y_g = c_1 e^{-3x} + c_2 e^{-2x} + e^x + x + 1}$$

QUESTION: HOW DID I KNOW TO MAKE $y_p = Ae^x + Bx + C$? DO I HAVE TO MEMORIZE THE BOX ON PAGE 200 ? IS THERE SOME UNIFYING UNDERLYING LOGIC ? YES! SEE THE ANNIHILATOR METHOD, IT HELPS US CHOOSE y_p. IT CALCULATES THE "S" FACTOR IN THE TEXT.

$\boxed{E73}$ $y'' = x^2$ with $y(0) = 0$ and $y'(0) = 1$

$\lambda^2 = 0 \Rightarrow y_h = c_1 + c_2 x$ so it $\underbrace{\text{"overlaps and } s = 2."}$ using text's terminology, pg. 200

$$y_p = x^2[Ax^2 + Bx + C] = Ax^4 + Bx^3 + Cx^2$$
$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx$$
$$y_p'' = 12Ax^2 + 6Bx + 2C$$

$$y_p'' = 12Ax^2 + 6Bx + 2C = x^2 \quad (\text{Subst. } y_p'' \text{ into DEq}^{\underline{n}})$$

Compare coefficients to obtain,

$$\begin{array}{ll} x^2] & 12A = 1 \\ x] & 6B = 0 \\ 1] & 2C = 0 \end{array} \Rightarrow \begin{array}{l} A = 1/12 \\ B = 0 \\ C = 0 \end{array} \Rightarrow y_p = \frac{1}{12}x^4$$

Thus the general sol$^{\underline{n}}$ is $y = c_1 + c_2 x + \frac{1}{12}x^4$. But, we don't want the general sol$^{\underline{n}}$ we wish to find the specific sol$^{\underline{n}}$ that satisfies the intial data,

$$\begin{array}{l} y(0) = c_1 = 0 \\ y'(0) = c_2 + \frac{1}{3}(0) = 1 \end{array} \Rightarrow \boxed{y = x + \frac{1}{12}x^4}$$

<u>Remark:</u> we could have found the general sol$^{\underline{n}}$ by integrating twice directly.

$$\int \frac{d}{dx}\frac{d}{dx}(y)\,dx = \int x^2 dx \Rightarrow \frac{dy}{dx} = \frac{1}{3}x^3 + c_2$$

$$\int \frac{dy}{dx}\,dx = \int\left(\frac{1}{3}x^3 + c_2\right)dx \Rightarrow y = \frac{1}{12}x^4 + c_2 x + c_1$$

This example is special because there is no y or y' think about why we cannot just straight away integrate <u>most</u> DEq$^{\underline{n}}$'s (seeing this attempted on tests makes me grimace, unless of course it works)

<u>Remark:</u> Also notice we apply intial conditions to the general sol$^{\underline{n}}$, <u>not</u> just the auxillary sol^{n} alone.

<u>QUESTION:</u> HOW SHOULD WE ANTICIPATE $y_p = Ax^4 + Bx^3 + Cx^2$?
ANSWER, THE ANNIHILATOR METHOD AND/OR LOTS OF MEMORIZATION / EXPERIENCE.

Annihilator Method

In short, this gives a proof of why our guesses for Y_p work. Our goal is to transform a nonhomogeneous ODE $L[Y](x) = g(x)$ into a corresponding homogeneous eq$^{\underline{n}}$ $AL[Y](x) = 0$. We will then find Y_p and Y_h for the original eq$^{\underline{n}}$ residing in the general sol$^{\underline{n}}$ for $AL[Y](x) = 0$.

$\boxed{E74}$ $Y'' + 2Y' + Y = e^{-x}$ here $L = D^2 + 2D + 1 = (D+1)^2$

We note $(D+1)e^{-x} = -e^{-x} + e^{-x} = 0$ so this suggests we choose $A = D+1$ so that $AL[Y] = A[e^{-x}] = 0$, notice,

$$AL = (D+1)(D+1)^2 = (D+1)^3$$

That is $AL[Y] = 0$ has auxillary eq$^{\underline{n}}$ $(\lambda+1)^3 = 0$ so $\lambda = -1$ with multiplicity 3 hence $\boxed{Y = \underbrace{c_1 e^{-x} + c_2 x e^{-x}}_{\text{auxillary sol}^{\underline{n}} \text{ to } L[Y] = e^{-x}} + \underbrace{c_3 x^2 e^{-x}}_{\text{correct } Y_p \text{ guess, complete with the } x^2 \text{ to adjust for overlap.}}}$

- to complete the problem we'd subst. $c_3 x^2 e^{-x}$ back in to $L[Y] = e^{-x}$ and determine c_3.

$\boxed{E75}$ $Y'' = xe^x + \sin x$ here $L = D^2$ we need to find the annihilator A of $xe^x + \sin x$

$$(D^2+1)\sin x = -\sin x + \sin x = 0$$
$$(D-1)^2 xe^x = 0$$ (see page ___)

Hence $A = (D^2+1)(D-1)^2$ then $AL = D^2(D^2+1)(D-1)^2$ which means $AL[Y] = 0 \Rightarrow Y = \underbrace{c_1 + c_2 x}_{Y_h} + \underbrace{c_3 \cos x + c_4 \sin x + c_5 e^x + c_6 x e^x}_{\text{correct guess for } Y_p}$

Remark: $\boxed{E2}$ illustrates the superposition principle. The total guess for $Y_p = Y_{p_1} + Y_{p_2}$ where $Y_{p_1} = c_3 \cos x + c_4 \sin x$ stems from $g_1(x) = \sin(x)$ and $Y_{p_2} = c_5 e^x + c_6 x e^x$ stems from $g_2(x) = xe^x$. In view of the superposition principle we can treat any nonhomogeneous term which is a sum and/or product of polynomials, exponentials or sines or cosines with the METHOD OF UNDET. COEFFICIENTS. What about $g(x) = \tan(x)$? We'll have to wait for variation of parameters.

E76 FIND Y_p VIA THE ANNIHILATOR METHOD.

$$(D^2+1)(D+3)^2 y = x^2 e^{-3x} + \cos(x)$$

Let's use the anihilator method. First notice the homogeneous sol$^{\underline{n}}$ has the form:

$$y_h = c_1 \cos x + c_2 \sin x + c_3 e^{-3x} + c_4 x e^{-3x}$$

[this follows quickly since we know that $P(D)\, e^{\lambda x} = P(\lambda) e^{\lambda x}$, the polynomial in D is the same as the aux. eq$^{\underline{n}}$. and

$$(\lambda^2+1)(\lambda+3)^2 = 0 \quad \text{has sol}^{\underline{n}}\text{'s } \lambda = \pm i \;\&\; \underset{\text{(twice)}}{\lambda = -3}$$
]

Now the anihilator of $x^2 e^{-3x} + \cos x$ can be found by thinking about what DEq$^{\underline{n}}$'s these functions arise as sol$^{\underline{n}}$'s to.

$x^2 e^{-3x}$ is sol$^{\underline{n}}$ to $(D+3)^3 Y = 0$

$\cos x$ is sol$^{\underline{n}}$ to $(D^2+1) Y = 0$

Thus $A = (D+3)^3 (D^2+1)$ will work. Now multiply by A our DEq$^{\underline{n}}$:

$$(D^2+1)^2 (D+3)^5 y = A(x^2 e^{-3x} + \cos(x)) = 0$$

This "corresponding homogeneous eq$^{\underline{n}}$" has the sol$^{\underline{n}}$

$$y = c_1 \cos(x) + c_2 \sin(x) + c_3 e^{-3x} + c_4 x e^{-3x} + \underbrace{c_5 x \cos x + c_6 x \sin x + c_7 x^2 e^{-3x} + c_8 x^3 e^{-3x} + c_9 x^4 e^{-3x}}_{\text{this is the correct choice for } y_p.}$$

- The other two methods would also have given this same results. Most of you need to think more about this.

E77 We can justify the Y_p formulas given below via the annihilator method. "Overlap" corresponds to repeated factor in AL which also appears in L.

3. a.) $y'' + 2y' + 10y = x^2 + x\cos(x)$

$Y_c = Y_h$

Complementary is same as homogeneous sol^n.

$\lambda^2 + 2\lambda + 10 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$

$\Rightarrow \boxed{Y_c = e^{-x}\left(c_1\cos(3x) + c_2\sin(3x)\right)}$

Then we guess

$\boxed{Y_p = Ax^2 + Bx + C + x\left(D\cos(x) + E\sin(x)\right) + F\cos(x) + G\sin(x)}$

no overlap so that'll do.

b.) $y'' + 2y' + y = e^{-x}$

$\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \quad \therefore \lambda = -1$ twice

$\Rightarrow \boxed{Y_c = c_1 e^{-x} + c_2 x e^{-x}}$

naively $Y_p^n = Ae^{-x}$ (overlaps)

less naive $Y_p^n = Axe^{-x}$ (still overlaps)

correct $\boxed{Y_p = Ax^2 e^{-x}}$

c.) $y'' + 9y = \cos(3x) - 6$

$\lambda^2 + 9 = 0 \therefore \lambda = \pm 3i \Rightarrow \boxed{Y_c = c_1\cos(3x) + c_2\sin(3x)}$

$Y_p^n = A\cos(3x) + B\sin(3x) + C$, naive, it overlaps Y_c.

$\boxed{Y_p = x\left(A\cos(3x) + B\sin(3x)\right) + C}$

• Notice Cx will not work.

d.) $y'' + 8y' + 12y = e^{-2x} + 7x$

$\lambda^2 + 8\lambda + 12 = (\lambda+2)(\lambda+6) = 0 \Rightarrow \lambda_1 = -2, \ \lambda_2 = -6$

$\Rightarrow \boxed{Y_c = c_1 e^{-2x} + c_2 e^{-6x}}$

$\boxed{Y_p = Axe^{-2x} + Bx + C}$ has no overlap, it'll work.

e.) $y'' + 16y = xe^x\sin(x)$

$\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i \Rightarrow \boxed{Y = c_1\cos(4x) + c_2\sin(4x)}$

think about differentiating $xe^x\sin(x)$. The x can goto 1 but the e^x survives and $\sin(x)$ becomes $\cos(x)$ thus

$\boxed{Y_p = e^x\left(A\sin(x) + B\cos(x) + x\left(C\sin(x) + D\cos(x)\right)\right)}$

clearly no overlap here.

$\boxed{E78}$ $y'' + y = \sin(x) \Rightarrow \lambda = \pm i \Rightarrow \quad y_c = C_1\cos(x) + C_2\sin(x)$

Notice if we try $y_p = A\cos(x) + B\sin(x)$ we'll find $y_p'' + y_p = 0 \neq \sin(x)$ So some other guess must be made for y_p. With II, in mind we'll try to multiply our original "naive guess" by x:

$$y_p = x(A\sin(x) + B\cos(x))$$

$$y_p' = A\sin(x) + B\cos(x) + x(A\cos(x) - B\sin(x))$$

$$y_p'' = A\cos(x) - B\sin(x) + A\cos(x) - B\sin(x) + \underbrace{x(-A\sin(x) - B\cos(x))}_{-y_p}$$

Now

$$y_p'' + 9y_p = 2A\cos(x) - 2B\sin(x) - y_p + y_p$$

$$= 2A\cos(x) - 2B\sin(x) = \sin(x)$$

Comparing coefficients: $A = 0$ & $B = -1/2$. So we find

$$\boxed{y = C_1\cos(x) + C_2\sin(x) - \frac{1}{2}x\cos(x)}$$

$\boxed{E79}$ $y'' + 8y' + 16y = 3x^2 - 2$ $\quad$ find gen. sol$^{\underline{n}}$

$\lambda^2 + 8\lambda + 16 = (\lambda+4)(\lambda+4) \therefore \lambda = -4 \therefore y_c = C_1e^{-4x} + C_2xe^{-4x}$

Now guess: $y_p = Ax^2 + Bx + C \longleftarrow$ (no overlap)

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Now Subst.

$$y_p'' + 8y_p' + 16y_p = 2A + 8[2Ax + B] + 16[Ax^2 + Bx + C]$$

$$= (2A + 8B + 16C) + x(16A + 16B) + x^2(16A) = 3x^2 - 2$$

Equate Coefficients:

x^2: $16A = 3 \quad \therefore \boxed{A = 3/16}$

x^1: $16(A+B) = 0 \quad \therefore B = -A = \boxed{-3/16 = B}$

x^0: $2A + 8B + 16C = -2 \Rightarrow 16C = -2 + 6A$

$$C = \frac{-2+6A}{16} = \frac{-2 + 18/16}{16} = \frac{-7}{128}$$

$$\boxed{y = C_1e^{-4x} + C_2xe^{-4x} + \frac{3}{16}x^2 - \frac{3}{16}x - \frac{7}{128}}$$

$\boxed{E80}$ $y'' + 9y = e^{2x}\sin(x)$

$\lambda^2 + 9 = 0 \;\therefore\; \lambda = \pm 3i \;\therefore\; y_c = c_1 \sin(3x) + c_2 \cos(3x)$

Guess $y_p = e^{2x}(A\sin(x) + B\cos(x))$

$y_p' = 2e^{2x}(A\sin(x) + B\cos(x)) + e^{2x}(A\cos(x) - B\sin(x))$

$= e^{2x}[(2A - B)\sin(x) + (2B + A)\cos(x)]$

$y_p'' = 2e^{2x}[(2A-B)\sin(x) + (2B+A)\cos(x)] + e^{2x}[(2A-B)\cos(x) - (2B+A)\sin(x)]$

$= e^{2x}[\sin(x)((4A-2B) - (2B+A)) + \cos(x)(4B+2A + 2A - B)]$

$= e^{2x}[\sin(x)(3A - 4B) + \cos(x)(3B + 4A)]$

Subst.

$y_p'' + 9y_p = e^{2x}[\sin(x)(3A-4B) + \cos(x)(3B+4A)] + 9e^{2x}(A\sin(x) + B\cos(x))$

$= e^{2x}[\sin(x)(12A - 4B) + \cos(x)(12B + 4A)] = e^{2x}\sin(x)$

Equating Coefficients:

$12A - 4B = 1$

$12B + 4A = 0 \Rightarrow A = -3B$

$\Rightarrow 12(-3B) - 4B = 1$

$\Rightarrow -40B = 1 \;\therefore\; B = -1/40 \;\therefore\; A = 3/40$

Hence $\boxed{y = c_1 \sin(3x) + c_2\cos(3x) + e^{2x}\left(\frac{3}{40}\sin(x) - \frac{1}{40}\cos(x)\right)}$

E81 $y'' + y = t\cos t$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h = c_1 \cos t + c_2 \sin t$

Thus there is overlap and multiplicity of i is 1.

$$y_p = t[At + B]\cos t + t[Ct + D]\sin t$$

$$(*) \quad y_p = (At^2 + Bt)\cos t + (Ct^2 + Dt)\sin t$$

$$y_p' = (2At + B)\cos t - (At^2 + Bt)\sin t + (2Ct + D)\sin t + (Ct^2 + Dt)\cos t$$

$$y_p' = (2At + B + Ct^2 + Dt)\cos t + (2Ct + D - At^2 - Bt)\sin t$$

$$y_p'' = (2A + 2Ct + D)\cos t - (2At + B + Ct^2 + Dt)\sin t + (2C - 2At - B)\sin t + (2Ct + D - At^2 - Bt)\cos t$$

$$(**) \quad y_p'' = (2A + 2Ct + D + 2Ct + D - At^2 - Bt)\cos t + (-2At - B - Ct^2 - Dt + 2C - 2At - B)\sin t$$

$$t\cos t = y_p'' + y_p = (\cancel{At^2} + \cancel{Bt} + 2A + 2Ct + D + 2Ct + D - \cancel{At^2} - \cancel{Bt})\cos t + (\cancel{Ct^2} + \cancel{Dt} - 2At - B - \cancel{Ct^2} - \cancel{Dt} + 2C - 2At - B)\sin t$$

To get the last eq$^{\underline{n}}$ I just substituted $(*)$ & $(**)$ into the DEq$^{\underline{n}}$. Now compare coefficients to find,

$\cos t$: $2A + 4Ct + 2D = t$

$\sin t$: $-4At - 2B + 2C = 0$

$t\cos t$: $4C = 1$

$\cos t$: $2A + 2D = 0$

$t\sin t$: $-4A = 0$

$\sin t$: $-2B + 2C = 0$

Hmm... many times I've jumped straight to the 4 eq$^{\underline{n}}$'s but breaking it into stages might organize better, whatever works for you. Anyway those eq$^{\underline{n}}$'s aren't hard to solve

$C = 1/4$ and $A = 0$ (almost obvious)

$D = 0$ and $B = 1/4$ (follow from line above)

Thus

$$y_p = \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t$$

Yielding general sol$^{\underline{n}}$,

$$\boxed{y = c_1\cos t + c_2\sin t + \frac{1}{4}t\cos t + \frac{1}{4}t^2\sin t}$$

Summary: Find y_h, Guess y_p, determine coefficients then assemble general sol$^{\underline{n}}$ as $y = y_h + y_p$.

Some conceptual foundations to our recent calculations

• <u>Comparing Coefficients</u>: obvious question, how can we do it? why is it valid? I'll illustrate by example, suppose that

$$\boxed{(A+B+C)x^2 + (B+C)x + C = 6x^2 + 3x + 1} \quad *$$

Howto find A, B, C? We compare coefficients and get

$$\begin{array}{l|lcl}
x^0 & C = 1 & \Rightarrow & C=1 \\
x^1 & B+C = 3 & \Rightarrow & B=2 \\
x^2 & A+B+C = 6 & \Rightarrow & A=3
\end{array}$$

But how did I know the eq^{n}'s above held given $*$?
Well, since $*$ holds for all values of x we can choose $x=0$ to find

$$(A+B+C)\cdot 0^2 + (B+C)\cdot 0 + C = 6(0)^2 + 3(0) + 1 \Rightarrow \boxed{C=1}$$

Ok, now differentiate $*$,

$$\boxed{(A+B+C)(2x) + B + C = 12x + 3} \quad *'$$

And evaluate $*'$ at $x=0$ to get

$$(A+B+C)(2\cdot 0) + B + C = 12(0) + 3 \Rightarrow \boxed{B + C = 3}$$

We've recovered the $x^0|$ and $x^1|$ eq^{n}'s. Now differentiate $*'$ to obtain

$$(A+B+C)\cdot 2 = 12 \Longrightarrow \boxed{A+B+C = 6}$$

So, we see that the fact $*$ holds for all x, and a little differentiation, will validate the eq^{n}'s we claimed to be true by "comparing coefficients".

• In fact, it is not hard to see the ideas above can prove equating coefficients for n^{th} degree polynomials we'd just have to differentiate n-times instead of 2.

More conceptual foundations

I have mentioned the idea of "linear independence" lets define it (for functions of a real variable x)

Def$^{\underline{n}}$/ $f(x)$ and $g(x)$ are <u>linearly</u> <u>dependent</u> if $f(x) = c\,g(x)$ for all x in $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. If we cannot write $f(x)$ as a constant multiple of $g(x)$ then we say $f(x)$ and $g(x)$ are linearly independent (L.I.)

<u>Proposition</u>: if $f(x)$ and $g(x)$ are linearly dependent then they have proportional slopes.

Pf/ just differentiate $f(x) = c\,g(x) \Rightarrow f'(x) = c\,g'(x)$.

$\boxed{E82}$ <u>Examples of Linear Independence / dependence</u>

1.) x is L.I. from x^2 notice $\frac{dx}{dx} = 1$ while $\frac{d}{dx}x^2 = 2x$

clearly 1 is not proportional to $2x$.
thus x cannot be linearly dep. on x^2
hence x & x^2 are L.I.

2.) $c_1 e^x$ and $c_2 e^x$ are linearly dependent $c_1 e^x = \left(\frac{c_1}{c_2}\right) c_2 e^x$.
(assume $c_1, c_2 \neq 0$)

3.) $c_1 e^x$ and $c_2 x e^x$ are L.I.

4.) $1, x, x^2, x^3, \dots, x^n$ are L.I. (pairwise)

5.) $\sin(x)$ and $\cos(x)$ are L.I. (think about the graphs.)

<u>Remark</u>: Given some set of linearly independent functions we can compare coefficients in an eq$^{\underline{n}}$ involving those functions.

Why $Y_g = Y_c + Y_p$ solves $aY'' + bY' + cY = g(x)$

The complementary sol$^{\underline{n}}$ Y_c (aka homogeneous sol$^{\underline{n}}$) has

$$\boxed{aY_c'' + bY_c' + cY_c = 0} \; *$$

Whereas the particular sol$^{\underline{n}}$ Y_p satisfies,

$$\boxed{aY_p'' + bY_p' + cY_p = g(x)} \; **$$

We claimed that $Y_c + Y_p$ is the general sol$^{\underline{n}}$, lets prove it. Remember $(f+g)' = f' + g'$ and $(cf)' = cf'$,

$$a(Y_c + Y_p)'' + b(Y_c + Y_p)' + c(Y_c + Y_p) =$$

$$= a(Y_c'' + Y_p'') + b(Y_c' + Y_p') + cY_c + cY_p$$

$$= \underbrace{aY_c'' + bY_c' + cY_c}_{*} + \underbrace{aY_p'' + bY_p' + cY_p}_{**}$$

$$= 0 + g(x)$$

$$= g(x)$$ which proves our claim, $Y_g = Y_c + Y_p$ solves $aY'' + bY' + cY = g(x)$.

<u>Remark</u>: You can think of Y_c as being necessary to encode intial conditions into the general sol$^{\underline{n}}$. Remember a $n^{\underline{th}}$ order ODEq$^{\underline{n}}$ will have n-arbitrairy constants $c_1, c_2, \dots, c_n$ in the general sol$^{\underline{n}}$. This corresponds to the fact that we need n-independent pieces of data to specify a sol$^{\underline{n}}$. We've seen this for $n=1$ and $n=2$ if you think about it.

(Once upon a time, this was a take-home problem.)

E83

(2) $$Y'' - 2Y' + Y = e^x + x\cos(2x) + x^4$$

i.) $\lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0 \Rightarrow Y_c = c_1 e^x + c_2 x e^x$

ii.) $$Y_p = \underbrace{Ax^2e^x}_{Y_{p_1}} + \underbrace{x(B\cos(2x) + C\sin(2x)) + D\cos(2x) + E\sin(2x)}_{Y_{p_2}} + \underbrace{Fx^4 + Gx^3 + \cdots}_{Y_{p_3}}$$

$$Y_{p_1}' = A(2x + x^2)e^x$$
$$Y_{p_1}'' = A(2 + 2x + 2x + x^2)e^x$$

$$Y_{p_1}'' - 2Y_{p_1}' + Y_{p_1} = A(x^2 + (2x + x^2)(-2) + 2 + 4x + x^2)e^x$$
$$= A(x^2 - 4x - 2x^2 + 2 + 4x + x^2)e^x$$
$$= A(2)e^x = e^x \Rightarrow A = 1/2$$

$$Y_{p_2}' = B\cos(2x) + C\sin(2x) + x(-2B\sin(2x) + 2C\cos(2x)) - 2D\sin(2x) + 2E\cos(2x)$$
$$Y_{p_2}' = \cos(2x)[B + 2Cx + 2E] + \sin(2x)[C - 2Bx - 2D]$$

$$Y_{p_2}'' = -2\sin(2x)[B + 2Cx + 2E] + \cos(2x)[2C] + 2\cos(2x)[C - 2Bx - 2D] + \sin(2x)[-2B]$$

$$Y_{p_2}'' = \cos(2x)[2C + 2C - 4Bx - 4D] + \sin(2x)[-2B - 4Cx - 4E - 2B]$$

$$x\cos(2x) = Y_{p_2}'' - 2Y_{p_2}' + Y_{p_2} = \cos(2x)[4C - 4D - 4Bx] + \sin(2x)[-4E - 4B - 4Cx] - 2\cos(2x)[B + 2E + 2Cx] - 2\sin(2x)[C - 2D - 2Bx] + \cos(2x)[Bx + D] + \sin(2x)[Cx + E]$$

$\underline{\cos(2x)}$ $4C - 4D - 4Bx - 2B - 4E - 4Cx + Bx + D = x$

$$x(-4B - 4C + B) + 4C - 3D - 2B - 4E = x$$
$$-3B - 4C = 1 \quad \& \quad 4C - 3D - 2B - 4E = 0$$

$\underline{\sin(2x)}$ $-4E - 4B - 4Cx - 2C + 4D + 4Bx + Cx + E = 0$

$$x(-4C + 4B + C) + (-4E + E - 4B - 2C + 4D) = 0$$
$$-3C + 4B = 0 \quad\quad -3E - 4B - 2C + 4D = 0$$

2 $$\left.\begin{aligned} -3B-4C&=1\\ 4C-3D-2B-4E&=0\\ -3C+4B&=0\\ -3E-4B-2C+4D&=0\end{aligned}\right\} \sim \left[\begin{array}{cccc|c} -3 & -4 & 0 & 0 & 1\\ -2 & 4 & -3 & -4 & 0\\ 4 & -3 & 0 & 0 & 0\\ -4 & -2 & 4 & -3 & 0\end{array}\right]$$ augmented coefficient matrix

enter the matrix above into TI and use "rref" to find,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3/25\\ 0 & 1 & 0 & 0 & -4/25\\ 0 & 0 & 1 & 0 & -22/125\\ 0 & 0 & 0 & 1 & 4/125\end{array}\right] \Leftrightarrow \boxed{\begin{aligned} B&=-3/25\\ C&=-4/25\\ D&=-22/125\\ E&=4/125\end{aligned}}$$

Next,

$$y_{p_3} = Fx^4 + Gx^3 + Hx^2 + Ix + J$$
$$y_{p_3}' = 4Fx^3 + 3Gx^2 + 2xH + I$$
$$y_{p_3}'' = 12Fx^2 + 6Gx + 2H$$

$$\begin{aligned} x^4 = y_{p_3}'' - 2y_{p_3}' + y_{p_3} &= 12Fx^2 + 6Gx + 2H\\ &\quad - 8Fx^3 - 6Gx^2 - 4Hx - 2I\\ &\quad + Fx^4 + Gx^3 + Hx^2 + Ix + J\\ &= 2H - 2I + J\\ &\quad + x(6G - 4H + I)\\ &\quad + x^2(12F - 6G + H)\\ &\quad + x^3(-8F + G)\\ &\quad + x^4(F)\end{aligned}$$

Equating Coefficients:

$$F = 1$$
$$-8 + G = 0 \quad \therefore \; G = 8$$
$$12 - 48 + H = 0 \quad \therefore \; H = 36$$
$$48 - 144 + I = 0 \quad \therefore \; I = 96$$
$$72 - 192 + J = 0 \quad \therefore \; J = 120$$

$$\boxed{\begin{aligned} Y = c_1e^x + c_2xe^x + \tfrac{1}{2}x^2e^x + x\left(\tfrac{-3}{25}\cos(2x) - \tfrac{4}{25}\sin(2x)\right) - \tfrac{22}{125}\cos(2x) + \tfrac{4}{125}\sin(2x)\\ + x^4 + 8x^3 + 36x^2 + 96x + 120\end{aligned}}$$

~~Clearly I was incorrect about how to set up x^4, needed I & J.~~