VARIATION OF PARAMETERS (JUST $n=2$)

We study $\boxed{aY'' + bY' + cY = g(x)}$ Eq^{n}(1) which has fundamental sol^{n} set $\{y_1, y_2\}$. We suppose that there exists a sol^{n} of the form

$$\boxed{Y_p = V_1 y_1 + V_2 y_2} \quad \text{Eq}^{n}(2)$$

Let us <u>derive</u> conditions on V_1 and V_2 to insure that Eq^{n}(2) does indeed provide a sol^{n} to Eq^{n}(1).

$$Y_p' = V_1' y_1 + V_1 y_1' + V_2' y_2 + V_2 y_2' \underset{\text{(using constraint)}}{=\!=} V_1 y_1' + V_2 y_2'$$

We impose the <u>constraint</u> $y_1 V_1' + y_2 V_2' = 0$ so that we will not get V_1'' and V_2'' in Y_p''. You might ask what right do we have to <u>add</u> this constraint? The answer is just that we are looking for a sol^{n} that works, so if this added constraint doesn't get in the way of that then we're good to go. You'll notice the text is not to verbose on this point. I think in general something is probably lost by adding this constraint, however for a large class of problems the method works. A more fundamental question to ask is why should Eq^{n}(2) be the only form for Y_p to take, maybe Y_p cannot be encapsulated by that ansatz either. My musing aside lets continue,

$$Y_p'' = V_1' y_1' + V_1 y_1'' + V_2' y_2' + V_2 y_2''$$

Thus,

$$\begin{aligned} g &= aY_p'' + bY_p' + cY_p \\ &= a(V_1' y_1' + V_1 y_1'' + V_2' y_2' + V_2 y_2'') + b(V_1 y_1' + V_2 y_2') + c(V_1 y_1 + V_2 y_2) \\ &= V_1 \underbrace{(ay_1'' + by_1' + cy_1)}_{= 0} + V_2 \underbrace{(ay_2'' + by_2' + cy_2)}_{= 0} + a(V_1' y_1' + V_2' y_2') \end{aligned}$$

(y_1 & y_2 solve the aux. eq^{n})

The calculations on this page reveal that $Y_p = V_1 y_1 + V_2 y_2$ will solve Eq^{n}(1) provided V_1 & V_2 satisfy

$$\boxed{\begin{aligned} 0 &= y_1 V_1' + y_2 V_2' \\ g/a &= y_1' V_1' + y_2' V_2' \end{aligned}} \quad \text{Eq}^{n}(9) \xrightarrow[\text{calculation}]{\text{short}} \boxed{\begin{aligned} V_1 &= \int \frac{-gy_2}{y_1 y_2' - y_2 y_1'}\,dx \\ V_2 &= \int \frac{gy_1}{y_1 y_2' - y_2 y_1'}\,dx \end{aligned}} \quad \text{Eq}^{n}(10)$$

where $\{y_1, y_2\}$ are the fundamental sol^{n} set.

E84

$$y'' + 4y = \tan(2x)$$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow Y_1 = \cos 2x, \quad Y_2 = \sin 2x$$

Note, $W[Y_1, Y_2] = Y_1 Y_2' - Y_2 Y_1'$

$$= (\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2$$

Use eq^{n}(10),

$$V_1 = \int \frac{-\tan(2x)\sin(2x)}{Y_1 Y_2' - Y_2 Y_1'}\,dx = \int \frac{-\sin^2 2x}{2\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \frac{1 - \cos^2 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \sec(2x) + \frac{1}{2}\int \cos(2x)\,dx$$

$$= \frac{-1}{4}\ln|\sec(2x) + \tan(2x)| + \frac{1}{4}\sin(2x).$$

$$V_2 = \int \frac{\tan(2x)\cos(2x)}{2}\,dx = \int \frac{\sin(2x)}{2}\,dx$$

$$= \frac{-1}{4}\cos(2x)$$

Recall that $y_p = y_1 V_1 + y_2 V_2$ so assemble the general sol^{n},

$$\boxed{y = c_1\cos(2x) + c_2\sin(2x) - \frac{1}{4}\cos(2x)\ln|\sec(2x) + \tan(2x)|}$$

$$+ \cancel{\frac{1}{4}\cos(2x)\sin(2x)} - \cancel{\frac{1}{4}\sin(2x)\cos(2x)}$$