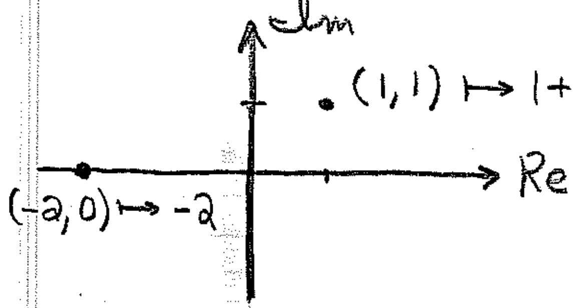
# Complex Variables : A Short Introduction

A complex variable is a variable whose values reside in the complex numbers. We denote the complex numbers by $\mathbb{C}$. If $z$ is a complex number then we say $z \in \mathbb{C}$. Every complex number has a real & imaginary part, $a, b \in \mathbb{R}$

$$z = a + ib \quad \Rightarrow \quad \mathrm{Re}\{z\} = a \;\; \& \;\; \mathrm{Im}\{z\} = b$$

This shows how $z \in \mathbb{C}$ can be identified as a point in a plane; $z \longmapsto (\mathrm{Re}\{z\}, \mathrm{Im}\{z\}) = (a, b)$. This is the complex plane, it represents complex numbers.




**Remark:** a complex # is also a 2-dimensional vector.

Usually an eq$^n$ involving complex variables has two real variable eq$^n$'s hidden within it. For example suppose we have the complex eq$^n$, $z^2 = z$ where $z = x + iy$, here $\mathrm{Re}\{z\} = x$ & $\mathrm{Im}\{z\} = y$. Both $x$ & $y$ are real variables. Consider then,

$$\begin{aligned} z^2 = z \quad &\Rightarrow \quad (x+iy)^2 = x + iy \\ &\Rightarrow \quad x^2 + 2ixy + i^2 y = x + iy \\ &\Rightarrow \quad x^2 - y^2 + i(2xy) = x + iy \quad \leftarrow (*) \\ &\Rightarrow \quad \underbrace{x^2 - y^2 = x}_{\text{Real Part of the Eq}^n\ (*)} \quad \& \quad \underbrace{2xy = y}_{\text{Imaginary Part of the Eq}^n\ (*)} \end{aligned}$$

I just used $i^2 = -1$, next lets collect all the basic rules for complex arithmatic & algebra,

# PROPERTIES AND DEFINITIONS

Suppose $z = x + iy$ and $w = a + ib$ where $x, y, a, b$ are real variables,

1. $zw = (x+iy)(a+ib) \equiv (xa - yb) + i(xb + ya)$
2. $z^* \equiv x - iy$
3. $zw = wz$
4. If $z \neq 0$ then $\frac{1}{z} = \frac{x - iy}{x^2+y^2}$.
5. $1/i = -i$
6. $z^* z = x^2 + y^2$

$\equiv$ means it's definition

Let me prove 4, we need $z\left(\frac{1}{z}\right) = 1$.

$$z\left(\frac{1}{z}\right) = (x+iy)\left(\frac{x-iy}{x^2+y^2}\right) = \frac{x^2 - \cancel{ixy} + \cancel{ixy} - i^2 y}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1.$$

we need $z \neq 0$ to insure that $x^2+y^2 \neq 0$.

Example: find $\frac{1}{1+i}$. Basically we just follow 4,

$$\frac{1}{1+i} = \frac{1-i}{1^2+1^2} = \frac{1-i}{2}.$$ In other words, $(1+i)^{-1} = \frac{1}{2} - \frac{i}{2}$

Now lets prove 6,
$$z^* z = (x - iy)(x+iy)$$
$$= x^2 + i\cancel{xy} - \cancel{iyx} - i^2 y$$
$$= x^2 + y^2. //$$

Notice we could also write the reciprocal in terms of $z, z^*$,

$$\frac{1}{z} = \frac{z^*}{z^* z}$$

the operation "$*$" is called complex conjugation it has many nice properties, $(z+w)^* = z^* + w^*$ and $(zw)^* = z^* w^*$.

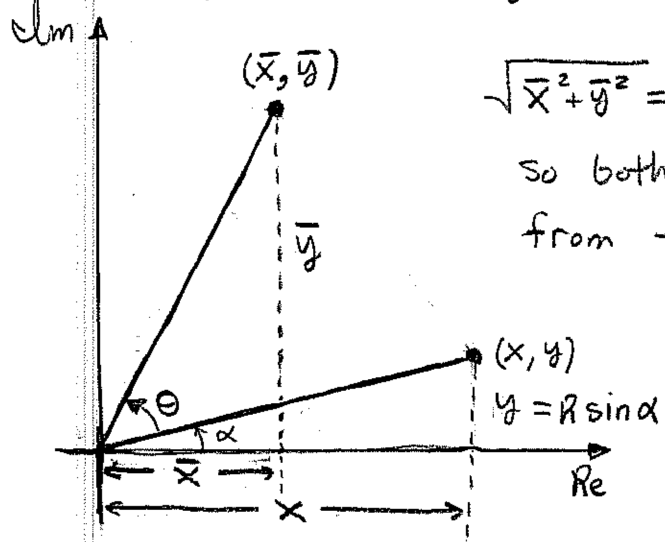
Upto now these ideas should be review from highschool. What follows is more useful and is likely new to you.

$$\boxed{\text{Euler's Identity}: \quad e^{i\theta} = \cos\theta + i\sin\theta}$$

Let me attempt to ellucidate the geometric foundations of this expression. Lets consider a point $z = x+iy$, for graphical convenience take $x, y > 0$. Consider,

$$
\begin{aligned}
e^{i\theta} z &= (\cos\theta + i\sin\theta)(x+iy) \\
&= \cos\theta\, x - \sin\theta\, y + i(\sin\theta\, x + \cos\theta\, y) \\
&= \bar{x} + i\bar{y}
\end{aligned}
$$

where I've defined $\bar{x} = x\cos\theta - y\sin\theta$ & $\bar{y} = x\sin\theta + y\cos\theta$ if you had studied rotations in the plane before these would be familar, but in case you haven't lets draw the picture




$\sqrt{\bar{x}^2+\bar{y}^2} = \sqrt{(\cos\theta x - \sin\theta y)^2 + (\sin\theta x + \cos\theta y)^2} = \sqrt{x^2+y$

So both $(x,y)$ & $(\bar{x},\bar{y})$ are distance $R = \sqrt{x^2+y$ from the origin. I let "$\alpha$" be th standard angle relative to the Re axis in the CCW direction. its easy to see that

$x = R\sin\alpha$ & $y = R\cos\alpha$

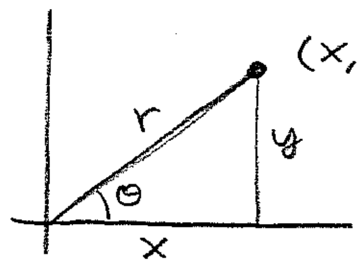
whereas clearly $(\bar{x},\bar{y})$ is at $\theta+\alpha$ so $\bar{x} = R\sin(\alpha+\theta)$ & $\bar{y} = R\cos(\alpha+\theta)$

$$
\begin{aligned}
\bar{x} &= R\sin(\alpha+\theta) = R\sin\alpha\cos\theta + R\cos\alpha\sin\theta = y\cos\theta + x\sin\theta \\
\bar{y} &= R\cos(\alpha+\theta) = \underbrace{R\cos\alpha\cos\theta - R\sin\alpha\sin\theta}_{\text{adding angles formulas for sin \& cos.}} = x\cos\theta - y\sin\theta
\end{aligned}
$$

Thus multiplying by $e^{i\theta} = \cos\theta + i\sin\theta$ rotates the point by $\theta$.

## Polar Form of Complex Number

Given $z = x + iy$ we can use $e^{i\theta} = \cos\theta + i\sin\theta$ to rewrite $z$ as $z = re^{i\theta}$ where $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$.




$(x, y) = (r\cos\theta, r\sin\theta) \mapsto re^{i\theta}$.

so I used the adding angles trig. identities to help this identifaction make sense. However, we can take another perspective, assume $e^{i\theta} = \cos\theta + i\sin\theta$ then derive all sorts of identities from this simple fact. Well we also assume a few other properties to start,

## Properties of exp of Complex Number

Let $z = x + iy$ and $W = a + ib$ then

1. $e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y$
2. $e^{z+W} = e^z e^W$
3. for $n \in \mathbb{R}$ $(e^z)^n = e^{nz}$

with these few simple rules we'll be able to derive just about any trig. identity we could possibly need.

Notice: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

Adding & subtracting yields two formulas worth remembering

| | |
|---|---|
| $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ | $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ |

# DERIVING TRIGONOMETRIC IDENTITIES

I'll proceed by example. Keep the previous page in mind,

Example:

$$\cos^2\theta + \sin^2\theta = \left[\tfrac{1}{2}(e^{i\theta}+e^{-i\theta})\right]^2 + \left[\tfrac{1}{2i}(e^{i\theta}-e^{-i\theta})\right]^2$$

$$= \tfrac{1}{4}\left(\cancel{e^{2i\theta}} + e^{i\theta}e^{-i\theta} + e^{-i\theta}e^{i\theta} + \cancel{e^{-2i\theta}}\right)$$
$$- \tfrac{1}{4}\left(\cancel{e^{2i\theta}} - e^{i\theta}e^{-i\theta} - e^{-i\theta}e^{i\theta} + \cancel{e^{-2i\theta}}\right)$$

note $e^{i\theta}e^{-i\theta} = e^0 = 1$

$$= \tfrac{1}{4}(1+1) - \tfrac{1}{4}(-1-1) = \tfrac{4}{4} = 1.$$

Now this not that surprising, hopefully you already knew this one. How about $\sin(2\theta) = 2\sin\theta\cos\theta$ ?

Example:

$$2\sin\theta\cos\theta = 2\,\tfrac{1}{2i}(e^{i\theta}-e^{-i\theta})\,\tfrac{1}{2}(e^{i\theta}+e^{-i\theta})$$
$$= \tfrac{1}{2i}\left(e^{2i\theta} + 1 - 1 - e^{-2i\theta}\right)$$
$$= \tfrac{1}{2i}\left(e^{2i\theta} - e^{-2i\theta}\right) = \sin(2\theta).$$

Do you know another way to derive this fact? How about

Example:

$$\cos^2\theta = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta})\,\tfrac{1}{2}(e^{i\theta}+e^{-i\theta})$$
$$= \tfrac{1}{4}\left(e^{2i\theta} + 2 + e^{-2i\theta}\right)$$
$$= \tfrac{1}{2}\left(1 + \tfrac{1}{2}(e^{2i\theta}+e^{-2i\theta})\right)$$
$$= \tfrac{1}{2}\left(1+\cos(2\theta)\right).$$

**Example:**

$$\sin(\theta)\cos(2\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right)$$

$$= \frac{1}{4i}\left(e^{3i\theta} + e^{-i\theta} - e^{i\theta} - e^{-3i\theta}\right)$$

$$= \frac{1}{2}\frac{1}{2i}\left(e^{3i\theta} - e^{-3i\theta}\right) - \frac{1}{2}\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= \frac{1}{2}\sin(3\theta) - \frac{1}{2}\sin(\theta).$$

**Example:**

$$\sin^2\theta\cos^2\theta = \left[\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right]^2\left[\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right]^2$$

$$= \frac{-1}{16}\left[e^{2i\theta} - 2 + e^{-2i\theta}\right]\left[e^{2i\theta} + 2 + e^{-2i\theta}\right]$$

$$= \frac{-1}{16}\Big[e^{4i\theta} + \cancel{2e^{2i\theta}} + 1$$
$$- \cancel{2e^{2i\theta}} - 4 - \cancel{2e^{-2i\theta}}$$
$$+ 1 + \cancel{2e^{-2i\theta}} + e^{-4i\theta}\Big]$$

$$= \frac{-1}{16}\left[e^{4i\theta} + e^{-4i\theta}\right] + \frac{1}{8}$$

$$= -\frac{1}{8}\cos(4\theta) + \frac{1}{8}.$$

*Now there are often ways of combining old trig. identities to get new ones, my point here is you can also choose a less clever route & just grind em' out via the imaginary exponentials.* A more clever sol would be,

$$\sin^2\theta\cos^2\theta = (1-\cos^2\theta)\cos^2\theta$$
$$= \cos^2\theta - \cos^4\theta$$
$$= \frac{1}{2}(1+\cos 2\theta) - \frac{1}{4}(1+\cos(2\theta))^2$$
$$= \frac{1}{2}(1+\cancel{\cos 2\theta}) - \frac{1}{4}(1 + \cancel{2\cos 2\theta}) - \frac{1}{4}\cos^2(2\theta)$$
$$= \frac{1}{4} - \frac{1}{8}(1-\cos(4\theta)) = \frac{1}{8} - \frac{1}{8}\cos$$

## ADDING ANGLES FORMULAS

These follow from an indirect argument (the direct argument is harder),

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b)$$

On the other hand we can use laws of exponents,

$$\begin{aligned} e^{i(a+b)} &= e^{ia}\, e^{ib} \\ &= (\cos a + i\sin a)(\cos b + i\sin b) \\ &= (\cos a \cos b - \sin a \sin b) + i(\cos a \sin b + \sin a \cos b) \end{aligned}$$

Therefore we find that,

$$\cos(a+b) + i\sin(a+b) = (\cos a \cos b - \sin a \sin b) + i(\cos a \sin b + \sin a \cos b)$$

Then we can read <u>two</u> real eq$^{n}$'s from this,

$$\boxed{\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(a+b) &= \cos(a)\sin(b) + \sin(a)\cos(b) \end{aligned}}$$

Given these we can derive the formula for $\tan(a+b)$,

$$\begin{aligned} \tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} = \left(\frac{\cos(a)\sin(b) + \sin(a)\cos(b)}{\cos(a)\cos(b) - \sin(a)\sin(b)}\right) \cdot \left(\frac{\frac{1}{\cos a \cos b}}{\frac{1}{\cos a \cos b}}\right) \\ &= \boxed{\frac{\tan(b) + \tan(a)}{1 - \tan(a)\tan(b)} = \tan(a+b)} \end{aligned}$$

# De Moivre's Theorem

We take Euler's Identity $e^{i\theta} = \cos\theta + i\sin\theta$ and raise it to the $n^{th}$ power, notice $(e^{i\theta})^n = e^{ni\theta}$ so

$$e^{ni\theta} = \underline{\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n}$$

this is the theorem, it has many hidden treasures.

$\underline{n=2}$ $\quad \cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2$

$$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$$

Equating Re and Im parts we find two nice facts,

$$\boxed{\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta \end{aligned}}$$

$\underline{n=3}$ $\quad \cos(3\theta) + i\sin(3\theta) = (\cos\theta + i\sin\theta)^3$

$$\begin{aligned} &= (\cos\theta + i\sin\theta)(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta) \\ &= \cos^3\theta - \sin^2\theta\cos\theta + 2i\sin\theta\cos^2\theta \\ &\quad + i\sin\theta\cos^2\theta - i\sin^3\theta - 2\sin^2\theta\cos\theta) \\ &= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\sin\theta\cos^2\theta - \sin^3\theta) \end{aligned}$$

Equating Re & Im parts reveals another pair of identities,

$$\boxed{\begin{aligned} \cos(3\theta) &= \cos^3\theta - 3\sin^2\theta\cos\theta \\ \sin(3\theta) &= 3\sin\theta\cos^2\theta - \sin^3\theta \end{aligned}}$$

**Remark:** I'm not a big fan of this Th$^{\underline{m}}$, I think the direct approach yields identities of interest quicker. Anyway there are many other tricks but we'll content ourselves with those discussed thus far.

**Remark:** there is much more to say, I have bits & pieces scattered throughout the notes. Also I recommend the classic text by Churchill.

# USEFUL FACTS ABOUT OPERATORS AND COMPLEX NUMBERS

Consider again the operator $L = D - \lambda$. This operator acts on functions of $x$ as follows,

$$L[Y] = DY - \lambda Y$$
$$L[Y](x) = Y'(x) - \lambda Y$$

What function satisfies $L[Y] = 0$? I claim that $e^{\lambda x}$ will do the trick,

$$L[e^{\lambda x}](x) = \frac{d}{dx}\left(e^{\lambda x}\right) - \lambda e^{\lambda x} = \lambda e^{\lambda x} - \lambda e^{\lambda x} = 0.$$

This calculation certainly makes sense if $\lambda \in \mathbb{R}$, but in fact it also is reasonable for $\lambda \in \mathbb{C}$. We pause to discuss the meaning of $e^{\lambda x}$ when $\lambda \in \mathbb{C}$.

- We always assume $x$ is a real variable.

**Def$^n$/** Let $a, b \in \mathbb{R}$ and $a + ib \in \mathbb{C}$ where $i = \sqrt{-1}$

$$e^{a+ib} \equiv e^{a} e^{ib} \quad \text{where} \quad e^{ib} \equiv \cos b + i \sin b$$

these eq$^n$'s define what is meant by the exponential of a complex number

TERMINOLOGY: Functions which map $\mathbb{R} \longrightarrow \mathbb{C}$ are complex-valued functions of a real variable. For example,

$$\tilde{f}(x) = a(x) + i\, b(x)$$

Here $a, b$ are functions from $\mathbb{R} \longrightarrow \mathbb{R}$, I use the $\sim$ to emphasize $\tilde{f}(x)$ is complex.

- DIFFERENTIATING COMPLEX-VALUED FUNCTIONS

$$\frac{d}{dx}\left(\tilde{f}(x)\right) = \frac{d}{dx}\left(a(x) + i b(x)\right) = \frac{da}{dx} + i\frac{db}{dx}$$

$$\tilde{f}'(x) = a'(x) + i\, b'(x)$$

Just like differentiating $\vec{r}(t)$ from ma 242, we differentiate each component function ($\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$.)

# COMPLEX VALUED FUNCTIONS OF A REAL VARIABLE

Def$^{n}$/ $\tilde{f}(x) = a(x) + i\,b(x)$ has real and imaginary components which are both real-valued functions

$$\mathrm{Re}\{\tilde{f}\} = a \quad \text{and} \quad \mathrm{Im}\{\tilde{f}\} = b$$

Now lets go back and verify my claim that $(D-\lambda)e^{\lambda x} = 0$ even when $\lambda \in \mathbb{C}$. Let $\lambda = a+ib$ notice $\mathrm{Re}\{\lambda\} = a$ and $\mathrm{Im}\{\lambda\} = b$,

$$\begin{aligned}
(D-\lambda)e^{\lambda x} &= \left(\frac{d}{dx} - \lambda\right)e^{(a+ib)x} \\
&= \frac{d}{dx}\left[e^{ax}(\cos bx + i\sin bx)\right] - \lambda e^{\lambda x} \\
&= ae^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + bi\cos bx) - \lambda e^{\lambda x} \\
&= e^{ax}(a\cos bx + ia\sin bx - b\sin bx + ib\cos bx) - \lambda e^{\lambda x} \\
&= (a+ib)e^{ax}(\cos bx + i\sin bx) - \lambda e^{\lambda x} \\
&= \lambda e^{\lambda x} - \lambda e^{\lambda x} = 0 \quad \underline{\text{as claimed}}.
\end{aligned}$$

used $-1 = i \cdot i$

<u>Remark</u>: If you take complex variables you'll learn how to differentiate w.r.t. a complex variable, complex numbers for us are just an algebraic convenience we will ultimately be interested in real-valued functions of a real variable.

<u>FACT</u>: If $L[\tilde{y}](x) = 0$ then the real and imaginary parts are sol$^{n}$'s as well; $L[\mathrm{Re}\{\tilde{y}\}](x) = 0$ and $L[\mathrm{Im}\{\tilde{y}\}](x) = 0$. I've assumed that $L$ is a linear operator with property

$$L[\tilde{c}_1 y_1 + \tilde{c}_2 y_2] = \tilde{c}_1 L[y_1] + \tilde{c}_2 L[y_2]$$

for $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}$. When $L = D^n + P_1 D^{n-1} + \cdots + P_n$ we certainly have this property. ($L$ is "complex linear")

# ALGEBRA REFRESHER

Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients $a_{n-1}, \dots, a_2, a_1, a_0$ then the fundamental Th$^{\underline{m}}$ of algebra says $\exists$ n-roots say $\lambda_1, \lambda_2, \dots, \lambda_n$ that satisfy $P(\lambda_i) = 0$. These may be repeated and/or complex. This means we can factor

$$P(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_d)^{m_d}$$

Where $m_1 + m_2 + \cdots + m_d = n$. Now in the event some $\lambda_k = a+ib$ is a sol$^{\underline{n}}$ it follows that $\lambda_k^* = a-ib$ is likewise a sol$^{\underline{n}}$. That is complex roots come in conjugate pairs. This is a straight-forward consequence of the assumption the coefficients are real, if we didn't have the conjugate root then the factorization once multiplied out would give non-real coeff. on the other hand notice $(a, b \in \mathbb{R})$

$$\begin{aligned}(x-(a+ib))(x-(a-ib)) &= x^2 - x(a-ib) - (a+ib)x + (a+ib)(a-ib) \\ &= x^2 - 2ax + \cancel{ibx} - \cancel{ibx} + a^2 - \cancel{iba} + \cancel{iba} - i^2b \\ &= \underbrace{x^2 - 2ax + a^2 + b^2}_{\text{an irreducible quadratic.}}\end{aligned}$$

This is the beauty of conjugate pairs, they make the $i$ factors cancel out. Anyway this means that we can factor <u>any</u> polynomial with <u>real</u> coefficients into a bunch of real linear factors and irred. quad. factors (possibly repeated)

$$P(x) = \underbrace{(x-\lambda_1)^{m_1}\cdots(x-\lambda_r)^{m_r}}_{\lambda_1,\dots,\lambda_r \in \mathbb{R}} \underbrace{(x^2+B_1x+C_1)^{n_1}\cdots(x^2+B_sx+C_s)^{n_s}}_{\substack{B_1^2-4C_1, \dots, B_s^2-4C_2 < 0 \\ \text{irreducible quad. factors.}}}$$

With $n = m_1 + \cdots + m_r + 2n_1 + \cdots + 2n_s$.

- This is the factorization that most explicitly reveals the aspects of $P(x)$ we need to use.

$\lambda = a+ib$
$\lambda^* = a-ib$

FIND REAL Sol$^n$'s to $(D-\lambda)^n(D-\lambda^*)^n Y = 0$

We know that $(D-\lambda)^n Y = 0$ has sol$^n$'s
$\tilde{Y}_1 = e^{\lambda x}$, $\tilde{Y}_2 = xe^{\lambda x}$, ..., $\tilde{Y}_n = x^{n-1}e^{\lambda x}$
but $\lambda = a+ib$ so these are complex,

$$\tilde{Y}_1 = e^{ax}\cos bx + ie^{ax}\sin bx$$
$$\tilde{Y}_2 = xe^{ax}\cos bx + ixe^{ax}\sin bx$$
$$\vdots$$
$$\tilde{Y}_n = x^{n-1}e^{ax}\cos bx + ix^{n-1}e^{ax}\sin bx$$

Notice $(D-\lambda^*)^n Y^*$ likewise has sol$^n$'s with $\lambda^* = a-ib$

$$\tilde{Y}_1^* = e^{ax}\cos bx - ie^{ax}\sin bx$$
$$\vdots$$
$$\tilde{Y}_n = x^{n-1}e^{ax}\cos bx - ix^{n-1}e^{ax}\sin bx$$

This means that $(D-\lambda)^n$ and $(D-\lambda^*)^n$ both possess real sol$^n$

$$\boxed{\{e^{ax}\cos bx,\ e^{ax}\sin bx,\ \dots,\ x^{n-1}e^{ax}\cos bx,\ x^{n-1}e^{ax}\sin bx\}}$$

of which we can count $2n$-sol$^n$'s in total.

Summary: we have found how to assemble real sol$^n$'s in the complex case and we already saw how to find sol$^n$'s in the real case with repeats. So we now have all we need to detail the general sol$^n$ to.

$$Y^{(n)}(x) + P_1(x)Y^{(n-1)}(x) + \cdots + P_n(x)Y(x) = 0$$

- Although to be careful we ought to prove these sol$^n$'s we have found are indeed linearly indenpondent. You should know how to prove LI for specific examples, you could use the Wronskian to check LI since these functions are all sol$^n$'s to a common DEq$^n$.