H83

§5.1 #2 Consider $z = f(x, y)$ where $f(0,0) = f_x(0,0) = f_y(0,0) = 0$.

(a.) $\mathbf{X}(u,v) = \langle u, v, f(u,v)\rangle$

$$\mathbf{X}_u = \langle 1, 0, f_u\rangle$$
$$\mathbf{X}_v = \langle 0, 1, f_v\rangle$$

$$\mathbf{X}_u \times \mathbf{X}_v = \langle -f_u, -f_v, 1\rangle$$

$$\Rightarrow U(u,v) = \frac{-f_u U_1 - f_v U_2 + U_3}{\sqrt{1 + f_u^2 + f_v^2}}$$

(trade u for x and v for y to see text's answer)

As $f_u(0,0) = f_v(0,0) = 0 \Rightarrow U(0,0) = U_3$ hence $u_1 = U_1$, $u_2 = U_2$ are clearly $\perp$ to U_3 which indicates u_1, u_2 are tangent to surface at $(0,0,0)$.

(b.) $S(v) = -\nabla_v U$, $U = (g_1, g_2, g_3)$

$$S(v) = -\sum_{j=1}^{3} v[g_j]\, U_j$$

$$S_p(u_1) = \left(-\sum_{j=1}^{3} U_1[g_j]\, U_j\right)(p)$$

$$= -\sum_{j=1}^{3} \frac{\partial g_j}{\partial x}(0,0,0)\, U_j$$

$$= -\frac{\partial U}{\partial x}$$

$$= \underline{f_{xx}^{(p)} U_1 + f_{xy}^{(p)} U_2}.$$

the other terms from $\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1+f_x^2+f_y^2}}\right)$ vanish at $P = (0,0,0)$ given $f_x(0,0) = f_{xy}(0,0) = f_{yy}(0,0) = 0$.

Likewise,

$$S_p(u_2) = -\frac{\partial U}{\partial y}$$

$$= \underline{f_{xy}(0,0)\, U_1 + f_{yy}(0,0)\, U_2}.$$

<u>Remark</u>: when $v = U_1$ or $v = U_2$, $v = U_3$ the covariant derivative just reduces to plain old partial diff.

§5.1 #3 Continuing #2 where we found formulas which reveal for $z = f(x,y)$ with $\nabla f = \langle 0,0\rangle$ and $f(0,0) = 0$ the shape operator S has matrix:

$$[S] = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{bmatrix}$$

determine rank of S_p for $P = (0,0,0)$

(a.) $z = xy \hookrightarrow [S] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\underline{\text{rank}(S) = 2}$.

B.t.w. by Def^{n}(3.1) page 216 we can calculate the Gaussian curvature $K = \det[S] = \underline{-1 = K}$.

(b.) $z = 2x^2 + y^2 \hookrightarrow [S] = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \underline{\text{rank } S = 2}$.

$\underline{K = 8}$.

(c.) $z = (x+y)^2 = x^2 + 2xy + y^2$

$[S] = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow \text{rank}(S) = 1$.

$\underline{K = 0}$.

(d.) $z = xy^2$

$$[S] = \begin{bmatrix} 0 & 2y \\ 2y & 2x \end{bmatrix}\Bigg|_{x=y=0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

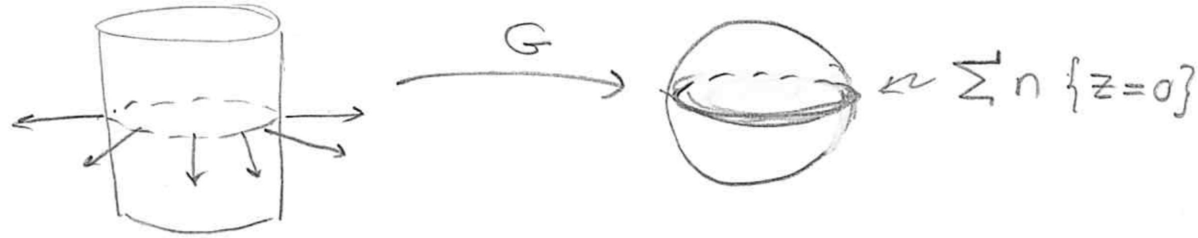
$\Rightarrow \underline{\text{rank}(S) = 0}$.

$K = 0$.

(H85)

§5.1 #4 Describe the image of the Gauss map $G : M \longrightarrow \Sigma$ where $G(P) = U(P)$ for

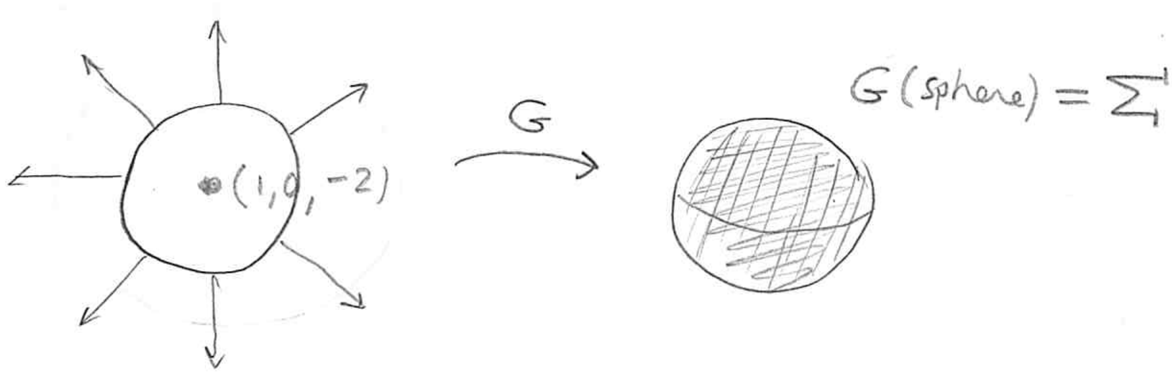
(a.) Cylinder: $x^2+y^2=r^2$



(b.) Cone: $z = \sqrt{x^2+y^2}$

(c.) Plane $x+y+z=0$

(d.) sphere
$(x-1)^2+y^2+(z+2)^2=1$



Remark: remember this when you contemplate §6.8 on total curvature. Note, $\text{area}(G(M)) = 0$ for $K=0$.

§5.3 #2 Given U_1, U_2 are orthonormal tangent vectors at $p \in M$ what geometric information follows from:

(a.) $S(U_1) \bullet U_2 = 0 \Rightarrow S(U_2) \bullet U_1 = 0 \Rightarrow [S] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$

that is, U_1 & U_2 are principle directions.

(b.) $S(U_1) + S(U_2) = 0 \Rightarrow [S] = \underbrace{\big[S(U_1) \big| -S(U_1) \big]}_{\text{rank}(S) \leq 1.}$

thus $K = 0$.

(c.) $S(U_1) \times S(U_2) = 0 \Rightarrow \underbrace{S(U_2) = k\, S(U_1)}_{\text{rank}(S) \leq 1.}$

thus $K = 0$

(d.) $S(U_1) \bullet S(U_2) = 0$

If $S(U_1), S(U_2) \neq 0$ then we have that σ is bending in $\perp$ directions for $\perp$ directions, however, it could be $S(U_1) = S(U_2) = 0$ so this means $S = 0$.

Remark: perhaps your sol^n's here will have more insight...

(H87)

§5.4#6 $M: z = \frac{x^2}{a^2} + \varepsilon \frac{y^2}{b^2}$ where $\varepsilon = \pm 1$

Find Gaussian curvature

$$\mathbf{X}(u,v) = (u, v, u^2/a^2 + \varepsilon(v^2/b^2))$$

$$\mathbf{X}_u = (1, 0, 2u/a^2) \qquad E = 1 + 4u^2/a^4$$

$$\mathbf{X}_v = (0, 1, 2\varepsilon v/b^2) \qquad F = 4\varepsilon uv/a^2b^2$$

$$G = 1 + 4\varepsilon v^2/b^4$$

$$U = \left(-\frac{2u}{a^2}, \frac{-2\varepsilon v}{b^2}, 1\right)/W, \quad W = \sqrt{1 + \frac{4u^2}{a^4} + \frac{4v^2}{b^2}}$$

$$\mathbf{X}_{uu} = (0, 0, 2/a^2) \qquad L = U \cdot \mathbf{X}_{uu} = 2/a^2W$$

$$\mathbf{X}_{uv} = (0, 0, 0) \qquad M = U \cdot \mathbf{X}_{uv} = 0$$

$$\mathbf{X}_{vv} = (0, 0, 2\varepsilon/b^2) \qquad N = U \cdot \mathbf{X}_{vv} = 2\varepsilon/b^2W$$

By Corollary 4.1

$$K = \frac{LN - M^2}{EG - F^2}$$

$$= \frac{\left(4\varepsilon/a^2b^2W^2\right) - 0}{(1 + 4u^2/a^4)(1 + 4\varepsilon v^2/b^4) - (16u^2v^2/a^4b^4)}$$

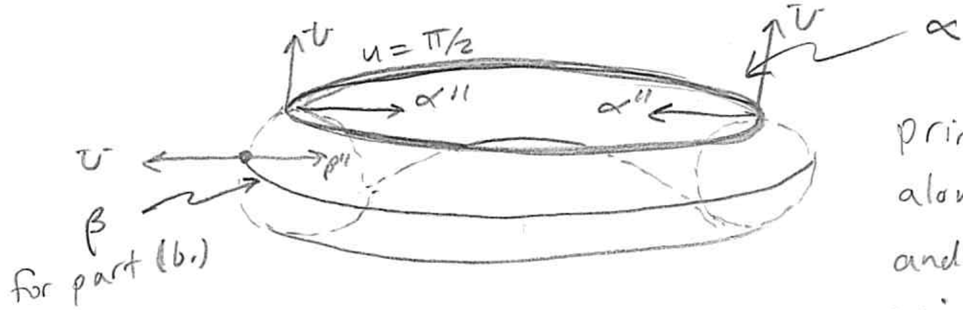
$$\Rightarrow \frac{4\varepsilon}{a^2b^2W^2}\left(\frac{1}{1 + 4u^2/a^4 + 4\varepsilon v^2/b^4 + 16u^2v^2/a^4b^4(\varepsilon - 1)}\right)$$

perhaps this simplifies...

I go on now...

§5.6 #2) To which of the three types — principal, asymptotic, geodesic — do the following curves belong?

(a.) top circle α of a torus



principal as $k_2 = 0$ along top of torus and α' points in that principal direction.

$$X(u,v) = ((R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r\sin(u))$$

fix $u = \pi/2$ $\quad \alpha(t) = (R\cos(t),\ R\sin t,\ r)$

$\alpha'(t) = (-R\sin t,\ R\cos t,\ 0)$, $\underbrace{\alpha''(t) = (-R\cos t, -R\sin t, 0)}$

clearly not normal to $M \Rightarrow$ α not geodesic

We also see α is asymptotic as $k(v) = S(v)\cdot v = 0$ for $v = \alpha'$. This is the case indicated in Lemma 6.4 (3)

"If $K(p) = 0$ then every direction is asymptotic if p is planar point; otherwise there is exactly one asympt. direction and it is also principal." (p. 242)

(b.) outer equator β of torus $(u = 0)$

$$\beta(t) = ((R+r)\cos t,\ (R+r)\sin t,\ 0)$$

$$\beta''(t) = (-(R+r)\cos t,\ -(R+r)\sin t,\ 0)$$

thus β is a geodesic as β'' is normal to torus.

$K > 0$ along β hence β not asymptotic.

It appears to me β is also principal in $\hat{e}_2$-direction.

(c.) left to reader for now ☺