Solve these problems on separate paper. One-sided solutions please.

Background: If G is a Lie group and H is a closed Lie subgroup, and G/H denotes the set of left-cosets of H. Then G/H is a manifold for which the quotient map of $G \to G/H$ is smooth. Moreover, if a space can be seen as G/H for appropriate G and H then G/H is a **homogeneous space**. The homogeneous space G/H naturally admits an action by G simply by left multiplication by elements of G. In particular, multiplication by elements of H produce no motion in G/H about $[e]$. It follows that if we are to view a given space S as a homogenous space then H should correspond to the set of motions on the space which fix points; that is, H should be the isotropy group of the space.

For example, if we consider the quotient $SO(n)/SO(n-1)$ we find that sphere S_{n-1} is a homogeneous space. Visualize the case for S_2: pick a point $p \in S_2$ note that any rotation about the p-axis leaves the point p fixed. More generally, any rotation in $SO(3)$ takes S_2 to itself once again and it can be argued the action of $SO(3)$ on S_2 is **transitive**. Identify that the subgroup of rotations about the p-axis is simply a copy of $SO(2)$ in $SO(3)$. In particular $SO(2)$ is the isotropy group of the sphere. It follows, $SO(3)/SO(2)$ is a model of S_2 as a homogeneous space. If there exists a Lie group which acts transitively on a space then it is possible to realize the space as a homogenous space. Not all spaces are homogeneous, read on Mathoverflow if you'd like to see why, it has to do with higher homotopy.

Of particular interest to this project are quotients formed from the affine versions of $SO(2)$ and $SO(3)$. You may recall $SO(n)$ provides rotations in $\mathbb{R}^n$. The group $ASO(n)$ provides rotations and translations in $\mathbb{R}^n$ which is often denoted $\mathbb{E}^n$ for **Euclidean n-space**. In particular, we put $SO(n)$ in a block and the translation vector in another: explicitly $ASO(n)$ are $(n+1) \times (n+1)$ matrices,

$$ASO(n) = \left\{ \begin{bmatrix} 1 & 0 \\ t & R \end{bmatrix} \mid t \in \mathbb{R}^n, \ R \in SO(n) \right\}$$

Then, a typical point has the form $(1, x) \in \mathbb{E}^n$ where $x \in \mathbb{R}^n$ and so $M(t, R) \in ASO(n)$ produces the following motion in euclidean space:

$$\begin{bmatrix} 1 & 0 \\ t & R \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ t + Rx \end{bmatrix}.$$

In this way we represent a rigid motion as a matrix group! In particular, we find $M(s, A)M(t, B) = M(s + At, AB)$. Furthermore, the quotient $ASO(n)/SO(n)$ is naturally identified with $\mathbb{E}^n$ (which is essentially $\mathbb{R}^n$ paired with Euclidean geometry).

Submanifolds of homogeneous space inherit structure from their context. In particular, we should like to study the problem of classifying if M and M' submanifolds of G/H are **equivalent**. In this context, the equivalence of M and M' is measured by the existence of $g \in G$ for which $M' = gM$. For example, two planes in $\mathbb{E}^3$ would be equivalent since we can find a rigid motion in $ASO(3)$ which maps a given plane to another. Furthermore, two curves with the same Frenet frame are congruent given our discussion earlier in this course. Once more, that ought to be seen in through the technique we attempt to describe here. The concept of **lifting** a function $f : M \to G/H$ to

$F : M \to G$ plays a significant role in this technique. If $\pi : G \to G/H$ is the quotient map then $f = \pi \circ F$. If F_1 and F_2 are two lifts of f then

$$F_2(x) = F_1(x)a(x)$$

for some function $a : M \to H$. We also have need of the Maurer-Cartan form on G. We define ω to be the unique left-invariant $\mathfrak{g}$-valued one-form on G for which $\omega_I : T_I G \to \mathfrak{g}$ is the identity. More to the point, we calculate the Maurer-Cartan form via:

$$\omega_a = g(a)^{-1} dg_a \qquad \text{a.k.a.} \qquad \omega = g^{-1} dg$$

where g is the mapping which embedds G as a particular real matrix subalgebra of $GL_d(\mathbb{R})$. The pull-back of ω onto the submanifold of interest reveals much about the geometric identity of M. However, beware there is a certain lift-dependence: for F_1, F_2 as above,

$$F_2^*(\omega) = a^{-1} F_1^*(\omega) a + a^{-1} da.$$

Note, for the problems we consider, G is typically $ASO(n)$ for $n = 2, 3$ so to define a lift from M to G it suffices to pick points paired with a frame of n-orthonormal vectors. That frame creates the needed $R \in SO(n)$ and hence the lift.

A beautiful identity exists for the Maurer-Cartan form which says the exterior derivative of ω can be calculated **algebraically**:

$$d\omega = -\omega \wedge \omega$$

Finally, we quote a theorem by Cartan from Theorem 1.6.10 in Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems, by Ivey and Landsberg:

> Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form ω. Let M be a manifold on which there exists a $\mathfrak{g}$-valued form ϕ satisfying $d\phi = -\phi \wedge \phi$. Then for any point $x \in M$ there exist a nbhd U of x and a map $f : U \to G$ such that $f^*\omega = \phi$. Moreover, any two such maps f_1, f_2 must satisfy $f_1 = L_a \circ f_2$ for some fixed $a \in G$.

Problem M1 Let $\{E_i \mid i = 1, \ldots, r\}$ serve as a basis for the real vector space $\mathfrak{g}$. Suppose $G = \exp(\mathfrak{g})$. Let $g(x^1, \ldots, x^r) = \exp(\sum x^i E_i)$. Note, the coordinate differential dx^i selects the coefficient of E_i in the following sense: if $A = \sum A^i E_i$ then $dx^j(A) = A^j$. **Show that $\omega_a = g(a)^{-1} dg_a$ is a $\mathfrak{g}$-valued one-form for which $\omega_I : T_I G \to \mathfrak{g}$ is the identity map.**

Problem M2 Calculate the Maurer-Cartan form for $SO(2)$. Notice $SO(2)$ is parametrized by θ. You can calculate ω in terms of $d\theta$ on $SO(2)$.

Problem M3 Calculate the Maurer-Cartan form for $ASO(2)$. Notice $ASO(2)$ is parametrized by x, y, θ. You can calculate ω in terms of dx, dy and $d\theta$ on $SO(2)$.

Problem M4 A typical $ASO(2)$ element has the form $g(x, y, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & \sin\theta \\ y & -\sin\theta & \cos\theta \end{bmatrix}$.Let $P_1 = \frac{\partial g}{\partial x}|_0$, $P_2 = \frac{\partial g}{\partial y}|_0$ and $J = \frac{\partial g}{\partial \theta}|_0$ where $|_0$ indicates setting $x = y = \theta = 0$. In this way we derive a basis $\{P_1, P_2, J\}$ for the Lie algebra $aso(2)$. Calculate

$$exp(xP_1 + yP_2 + \theta J)\exp(aP_1 + bP_2)$$

using the Baker-Campbell-Hausdorff relation upto just the first commutator correction.

Remark: in the quotient $ASO(2)/SO(2)$ the J-term is in some sense set to zero and the calculation above shows how the exponentials of P_1 and P_2 generate translations in the homogeneous space $\mathbb{R}^2$.

Problem M5 Propose an affine version of $SL(2, \mathbb{R})$. Elements of $ASL(2, \mathbb{R})$ would generate the motions such as $v \mapsto Av + b$ for $v \in \mathbb{R}^2$ for each choice $b \in \mathbb{R}^2$ and $A \in SL(2, \mathbb{R})$. Find a basis for the Lie algebra of the affine special linear group of order two.

Problem M6 Derive the structure equation $d\omega = -\omega \wedge \omega$. Hint: $gg^{-1} = I$ and remember $\omega = g^{-1}dg$ in the point-omitting notation. The derivation is much the same as we saw for Cartan's structure equations for $\mathbb{R}^3$. Understand, or begin to understand, that is just because $\mathbb{R}^3 = ASO(3)/SO(3)$ so the derivation we saw earlier is merely an adaptation of this much more general fact.

Problem M7 Suppose $C : \mathbb{R} \to M \subseteq \mathbb{R}^3$ is a regular, unit-speed, curve. Find a lift of C to $ASO(3)$ by assigning a point and frame for each $s \in \mathbb{R}$.

Problem M8 Suppose $F_1, F_2 : \mathbb{R} \to ASL(2, \mathbb{R})$ are lifts of regular, unit-speed, curves $\alpha_1, \alpha_2 : \mathbb{R} \to \mathbb{R}^2$ to $ASL(2)$. Given that $\mathbb{R}^2$ can be identified with $ASL(2, \mathbb{R})/SL(2, \mathbb{R})$ if $F_1^*(ds) = F^*(ds)$ then what can we say about α_1 and α_2?