# 1 Overview

The goal of these notes is to explain what Yang-Mills theory is both physically and mathematically. To begin we will outline how to formulate Electromagnetism as an abelian gauge theory. After that we'll generalize to the nonabelian gauge theory which is known as Yang-Mills theory. In the introductory sections we will content ourselves with a presentation which in the style of physics. Having established the equations that result from physical considerations we will then endeavor to restate these ideas in terms geometrically in a fiber bundle language.

# 2 Special Relativity and E&M

## 2.1 Maxwell's Equations

Electromagnetism is the study of how electric ($\vec{E}$) and magnetic ($\vec{B}$) fields influence matter. To begin, one studies how to solve Maxwell's equations in order to find the $\vec{E}$ and $\vec{B}$ given some particular set of sources, i.e. charge density $\rho$ and current density $\vec{J}$. One must solve the following set of coupled partial differential equations:

$$\begin{aligned} &\nabla \cdot \vec{E} = \rho && \nabla \cdot \vec{B} = 0 \\ &\nabla \times \vec{E} + \partial_t \vec{B} = 0 && \nabla \times \vec{B} - \partial_t \vec{E} = \vec{J} \end{aligned} \tag{1}$$

Physically this isn't quite right because we have omitted the permittivity ($\epsilon$) and the permeability ($\mu$) of the medium in which these equations are to be solved. We assume a linear medium so that these omitted objects are just constants which can again be put in after the mathematics is done. We will also set the speed of light to unity throughout. In fact we will be primarily interested in the behavior of the theory in free space, independent of any particular medium except spacetime itself.

## 2.2 Electric and Vector Potentials

The equations (1) are difficult to solve outright, but become more tractable thru the use of the electric potential ($\phi$) and the magnetic vector potential ($\vec{A}$). One reformulates Maxwell's equations above into an equivalent set of equations for the potentials. Typically these (Poisson) equations are easier to solve. Once solutions for the potentials are obtained then one can simply obtain $\vec{E}$ and $\vec{B}$ with some differentiation,

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla \phi - \partial_t \vec{A} \end{aligned} \tag{2}$$

Notice then that it is clear that the the choice of potentials is not unique. We can add, for an arbitrary function of spacetime $\lambda$, a constant function $\partial_t \lambda$ to $\phi$ and a gradient $-\nabla \lambda$ to $\vec{A}$ without changing the electric and magnetic fields $\vec{E}$ and $\vec{B}$.

$$\begin{aligned} \vec{B} &= \nabla \times (\vec{A} - \nabla \lambda) = \nabla \times \vec{A} \\ \vec{E} &= -\nabla(\phi + \partial_t \lambda) - \partial_t(\vec{A} - \nabla \lambda) = -\nabla \phi - \partial_t \vec{A} \end{aligned} \tag{3}$$

This freedom has been known for sometime, although originally it was thought to be just a quirk of the mathematics. We will delve deeper into these matters as we go on.

### 2.3 Special Relativity in a nutshell

Maxwell's equations suggest that the $\vec{E}$ and $\vec{B}$ are manifestations of a single object. More than this Maxwell's equations suggest that the speed of light is constant in any reference frame. This can be argued by studying the wave-equations for $\vec{E}$ and $\vec{B}$, one finds a speed independent of the coordinate used (see page 376 of Griffith's Introduction to Electrodynamics). This is in clear contradiction to many intuitive ideas about velocity addition. Einstein solved this dilemma by replacing *Newtonian* intuition with Special Relativity. Special relativity postulates that the speed of light is constant (all observers measure the same speed for a given light ray). Interestingly, although Einstein had to modify classical mechanics in a nontrivial way, Electromagnetism was already "covariant". That means that Electromagnetism intrinsically respects Special Relativity. We will prove explicitly that the equations of electromagnetism have the same form in any reference frame in which they are formulated. This is Einstein's requirement that the "laws" of physics should be the same in all inertial frames of reference. You are free to try to establish this covariance directly from Maxwell's equations, but there is a much better way as we shall see. Essentially, Special Relativity says that spacetime is locally $\mathbb{R}^4$ with the following metric (customarily we put time first then list the spatial coordinates $x^0 = t, x^1 = x, x^2 = y, x^3 = z$):

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{4}$$

The 4 dimensional space with the above metric is called Minkowski space. The $ds^2$ is called the interval, (or the square of the interval,but we will use the former). A "four-vector" is simply a vector in this space. We say the vector is "timelike" if it has positive interval, its "spacelike" if it has negative interval, and it is "lightlike" if is has zero interval. While I have written the metric in it's infinitesimal form it should be clear we can integrate $ds$ easily to find the finite form since $\eta_{\mu\nu}$ is the constant matrix.

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

The length of a vector $(V^\mu) = (V^0, V^1, V^2, V^3)$ in Minkowski space is defined to be

$$||V^\mu|| = \sqrt{|\eta_{\mu\nu} V^\mu V^\nu|} \tag{6}$$

The set of isometries of the metric above are called Lorentz transformations. These are linear transformations which leave the *length* of arbitrary four-vectors unchanged. One says that two coordinate systems $\{x^\mu\}$ and $\{\bar{x}^\nu\}$ are inertially related if there exists a Lorentz transformation connecting them. Given two inertially related coordinate systems we now calculate the general criteria that a linear transformation ($\Lambda$) must satisfy to be a Lorentz transformation, ( it is linear so $\bar{x}^\mu = \Lambda^\mu_\alpha x^\alpha$ from the start)

$$\begin{aligned} d\bar{s}^2 &= \eta_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu \\ &= \eta_{\mu\nu} d(\Lambda^\mu_\alpha x^\alpha) d(\Lambda^\nu_\beta x^\beta) \\ &= \eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta dx^\alpha dx^\beta \\ &= \eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu dx^\mu dx^\nu \\ &= ds^2 \end{aligned} \tag{7}$$

But we know that $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ thus we find that a Lorentz transformation must satisfy

$$\eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu = \eta_{\mu\nu} \tag{8}$$

The set of all such $\Lambda$ is the Lorentz group. If we further require that $\Lambda$ maintains the sign of the interval (takes spacelike to spacelike and timelike to timelike) then that smaller group of matrices is called the proper Lorentz group (SO(1,3)). These are the basic ingredients, we now recall how tensor components change under the Lorentz transformation $\bar{x}^\mu = \Lambda^\mu_\alpha x^\alpha$

$$\begin{aligned} &\bar{V}^\mu = \Lambda^\mu_\alpha V^\alpha \\ &\bar{V}_\alpha = (\Lambda^{-1})^\mu_\alpha V_\mu \\ &\bar{A}^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta A^{\alpha\beta} \\ &\bar{A}_{\alpha\beta} = (\Lambda^{-1})^\mu_\alpha (\Lambda^{-1})^\nu_\beta A_{\mu\nu} \end{aligned} \tag{9}$$

Furthermore it is convenient to recall that we can raise and lower indices with the metric $\eta_{\mu\nu}$. The formulae below remind us that in the presence of a metric a vector can be written "covariantly" (with its indices down) or "contravariantly" (with its indices up). The difference is somewhat artificial in that they contain the same data and are uniquely correlated thanks to the metric. We will make use of this and similar isomorphisms of higher rank tensors and their duals.

$$\begin{aligned} V^\mu &= \eta^{\mu\nu} V_\nu \\ V_\mu &= \eta_{\mu\nu} V^\nu \\ A^{\mu\nu} &= \eta^{\mu\alpha} \eta^{\nu\beta} A_{\alpha\beta} \\ A_{\mu\nu} &= \eta_{\mu\alpha} \eta_{\nu\beta} A^{\alpha\beta} \end{aligned} \tag{10}$$

We note that $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$ in general, in the case considered here we note $\eta^{\mu\nu} = \eta_{\mu\nu}$. Most of the ideas of this section can be more elegantly phrased in terms of some coordinate free geometric construction, we leave those for another place and time.

## 2.4 The Field Strength and 4-current of relativistic electrodynamics

Again I emphasize that Maxwell's equations were "relativistic" from the get-go. What we do now is simply a change of notation, nothing more physically speaking. To begin the charge density($\rho$) and current density ($\vec{J}$) can be grouped into a single 4-vector which we will call the current ($J^\mu$),

$$(J^\mu) = (\rho, \vec{J}) \quad a.k.a \quad (J_\mu) = (\rho, -\vec{J}) \tag{11}$$

Now the obvious question is what about the $\vec{E}$ and $\vec{B}$ fields? We note that Maxwell's equations suggest that these fields are manifestations of a single object. Thus we need to assemble them in a 6 component object, and an antisymmetric rank two tensor in four dimensions fits that exactly. We thus define the field strength $F_{\mu\nu}$ to be an antisymmetric tensor which contains the $\vec{E}$ and $\vec{B}$ fields as follows, note the convention $i, j = 1, 2, 3$ whereas $\mu, \nu = 0, 1, 2, 3$.

$$\begin{aligned} F_{0i} &= E_i \\ F_{ij} &= -\epsilon_{ijk} B_k \\ F_{\mu\mu} &= 0 \end{aligned} \tag{12}$$

Of course the last equation follows directly from the antisymmetry of the field tensor. Often it is convenient to view rank two tensor calculations in terms of matrix calculations, so it may be convenient at times to use the following matrix version of the field tensor,

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{13}$$

Or if you like we can raise the indices with the metric to yield,

$$(F^{\mu\nu}) = (\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{14}$$

While we are at it we might as well calculate the value of the scalar $F_{\mu\nu}F^{\mu\nu}$. We know it is a scalar because it has no free indices.( you can check this using eq. 9, you will get $\Lambda$ and $\Lambda^{-1}$ which cancel out, implicitly we assume that the field tensor is a tensor, we will prove this in a later section it's not hard )

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{0\nu}F^{0\nu} + F_{i\nu}F^{i\nu} \\ &= F_{00}F^{00} + F_{0j}F^{0j} + F_{i0}F^{i0} + F_{ik}F^{ik} \\ &= 2F_{i0}F^{i0} + F_{ij}F^{ij} \\ &= 2(-E_i)E_i + \epsilon_{ijk}B_k\epsilon^{ijl}B_l \\ &= -2\vec{E}\cdot\vec{E} + 2\delta^l_k B_k B_l \\ &= 2(\vec{B}\cdot\vec{B} - 2\vec{E}\cdot\vec{E}) \\ &= 2(B^2 - E^2) \end{aligned} \tag{15}$$

## 2.5 The 4-potential for relativistic electrodynamics

Next we note that the scalar potential $\phi$ and vector potential $\vec{A}$ can likewise be grouped into a single 4-potential we denote $A_\mu$

$$(A^\mu) = (\phi, \vec{A}) \quad a.k.a. \quad (A_\mu) = (\phi, -\vec{A}) \tag{16}$$

We note in passing that we can make the replacement $A^\mu \longrightarrow A^\mu + \partial^\mu \lambda$ and obtain the same $\vec{E}$ and $\vec{B}$ fields. This is the same statement as equation 3 in 4 dimensional language (note $(\partial_\mu) = (\partial_t, \nabla)$,whereas $(\partial^\mu) = (\partial_t, -\nabla)$). Previously we defined the field tensor in terms of the $\vec{E}$ and $\vec{B}$ fields, an equivalent definition of the field tensor is given by the 4 dimensional curl of the 4-potential namely,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{17}$$

It can be shown that this definition and the one given in terms of the $\vec{E}$ and $\vec{B}$ fields is in fact equivalent. One just needs to use equation (2) to see it explicitly. For example take the spatial part of the field tensor,

$$\begin{aligned} F_{ij} &= -\epsilon_{ijk} B_k \\ &= -\epsilon_{ijk} (\nabla \times \vec{A})_k \\ &= \epsilon_{ijk} \epsilon^{klm} \partial_l A_m \\ &= (\delta^{lm}_{ij} - \delta^{lm}_{ji}) \partial_l A_m \\ &= \partial_i A_j - \partial_j A_i \end{aligned} \tag{18}$$

Likewise we can show that,

$$\begin{aligned} F_{0i} &= E_i \\ &= (-\nabla\phi - \partial_t \vec{A})_i \\ &= -\partial_i \phi + \partial_t A_i \\ &= \partial_0 A_i - \partial_i A_0 \end{aligned} \tag{19}$$

This demonstrates that we can formulate the field tensor either directly in terms of the electric and magnetic fields, or in terms of the potential functions. This observation alone suggests that the field tensor should be invariant under the exchange $A_\mu \longrightarrow \bar{A}_\mu = A_\mu + \partial_\mu \lambda$. Afterall, $A_\mu$ and $\bar{A}_\mu$ lead to the same set of $\vec{E}$ and $\vec{B}$ fields. Lets see why,

$$\begin{aligned} \bar{F}_{\mu\nu} &= \partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= F_{\mu\nu} \end{aligned} \tag{20}$$

Crucial to this calculation is the assumption that the partial derivatives commute. This is of course true for inertial coordinates on Minkowski space.

## 2.6 Electromagnetism in action

Upto now we have relativistically reformulated the objects in Electromagnetism, however, it still remains to find how to restate Maxwell's equations in terms of these 4-dimensional objects. It is desirable to define physical theories by an action principal. One can show that symmetries of the action lead to symmetries of the equations of motion stemming from that action. Additionally, it will prepare us for the path integral formalism of Feynman which is defined using the action. Because we want a relativistic theory we need for the action to be a scalar with respect to Lorentz transformation (that is the action should be invariant under SO(1,3) rotations). A simple scalar to try is just $F^{\mu\nu}F_{\mu\nu}$. Define then the action (no sources yet, meaning $J^\mu = 0$) to be

$$S[A_\mu] = -\tfrac{1}{4}\int_{\mathbb{R}^4} F^{\mu\nu}F_{\mu\nu}d^4x \tag{21}$$

Notice while we have phrased the action in terms of the field tensor it should be understood the we take the potential $A_\mu$ to be the dynamical variable. The action is a functional that depends on the function $A_\mu$. As such we now vary the action with respect to the potential,

$$\begin{aligned}
\delta S &= -\tfrac{1}{4}\delta\int_{\mathbb{R}^4} F^{\mu\nu}F_{\mu\nu}d^4x \\
&= -\tfrac{1}{4}\delta\int_{\mathbb{R}^4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}F_{\mu\nu}d^4x \\
&= -\tfrac{1}{4}\int_{\mathbb{R}^4}\eta^{\mu\alpha}\eta^{\nu\beta}\big((\delta F_{\alpha\beta})F_{\mu\nu} + F_{\alpha\beta}(\delta F_{\mu\nu})\big)d^4x \\
&= -\tfrac{1}{2}\int_{\mathbb{R}^4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}(\delta F_{\mu\nu})d^4x \\
&= -\tfrac{1}{2}\int_{\mathbb{R}^4}F^{\mu\nu}(\delta F_{\mu\nu})d^4x \\
&= -\tfrac{1}{2}\int_{\mathbb{R}^4}F^{\mu\nu}\big(\delta(\partial_\mu A_\nu) - \delta(\partial_\nu A_\mu)\big)d^4x \\
&= -\tfrac{1}{2}\int_{\mathbb{R}^4}\big(-(\partial_\mu F^{\mu\nu})\delta A_\nu + (\partial_\nu F^{\mu\nu})\delta A_\mu\big)d^4x \\
&= -\tfrac{1}{2}\int_{\mathbb{R}^4}\big(-\partial_\mu F^{\mu\nu} + \partial_\mu F^{\nu\mu}\big)\delta A_\nu d^4x \\
&= \int_{\mathbb{R}^4}\big(\partial_\mu F^{\mu\nu}\big)\delta A_\nu d^4x \\
&= 0
\end{aligned} \tag{22}$$

Note that the step passing from the sixth to the seventh equality was integration by parts, we assume the total divergence contributes nothing by applying Stoke's theorem and assuming the locality of the fields. Because the equation above holds for arbitrary variations of the potentials it follows from the least action principal that we have the following equations of motion,

$$\partial_\mu F^{\mu\nu} = 0 \tag{23}$$

This amounts to 4 equations which give 2 of Maxwell's equations ( these two are commonly referred as the inhomogeneous equations although that is hard to see at the moment ), using $F^{\mu\mu} = 0$ we find

$$\begin{aligned}
\partial_\mu F^{\mu 0} &= \partial_i F^{i0} = \partial_i E_i = \nabla\cdot\vec{E} = 0 \\
\partial_\mu F^{\mu j} &= \partial_0 F^{0j} + \partial_i F^{ij} = -\partial_t E_j - \partial_i\epsilon_{ijk}B_k = (\nabla\times\vec{B} - \partial_t\vec{E})^j = 0
\end{aligned} \tag{24}$$

We used $\epsilon_{ijk}v^jw^k = (\vec{v}\times\vec{w})_i$ and the antisymmetry of $\epsilon_{ijk}$ in the last step above. We should emphasize these are the equations of pure gauge theory, the sources are all zero. That said let us put in the sources now. We know that $A_\mu$ describes the Electromagnetic response to the source $J^\nu$. We can modify the action by adding another scalar that couples the field to the sources, namely $J^\mu A_\mu$. Study then how this term behaves under a variation of the potential,

$$\delta(J^\mu A_\mu) = J^\mu \delta A_\mu \tag{25}$$

That was easy. Now lets define the action with sources,

$$S[A_\mu] = \int_{\mathbb{R}^4} \left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu\right) d^4x \tag{26}$$

Then the principal of least action $\delta S = 0$ yields the equations,

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{27}$$

This tensor equation corresponds to Maxwell's equations with sources:

$$\begin{aligned} &\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \partial_i E_i = \nabla \cdot \vec{E} = J^0 = \rho \\ &\partial_\mu F^{\mu j} = \partial_0 F^{0j} + \partial_i F^{ij} = -\partial_t E_j + \partial_i \epsilon_{ijk} B_k = (\nabla \times \vec{B} - \partial_t \vec{E})^j = J^j = (\vec{J})^j \end{aligned} \tag{28}$$

We then see how the inhomogeneous Maxwell's equations can be cleanly derived from an action principal. The other two homogeneous Maxwell's equations follow from the antisymmetry of the field tensor. Recall $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ thus

$$\begin{aligned} \nabla \cdot \vec{B} &= \partial_i B_i \\ &= \partial_i(\tfrac{1}{2}\epsilon_{ijk}F_{jk}) \\ &= \tfrac{1}{2}\epsilon_{ijk}\partial_i(\partial_j A_k - \partial_k A_j) \\ &= \tfrac{1}{2}(\epsilon_{ijk}\partial_i\partial_j A_k - \epsilon_{ijk}\partial_k\partial_i A_j) \\ &= \tfrac{1}{2}(\epsilon_{ijk}\partial_i\partial_j A_k - \epsilon_{jki}\partial_i\partial_j A_k) \\ &= \tfrac{1}{2}(\epsilon_{ijk} - \epsilon_{jki})\partial_i\partial_j A_k \\ &= 0 \end{aligned} \tag{29}$$

The remaining homogeneous Maxwell equation can likewise also be derived from the skew symmetry of the field tensor written in terms of the potential. It is interesting to note that we have not explicitly encountered the *Hodge dual*. Actually it is implicit within the action and we will see this more explicitly as we expound these ideas in coordinate free language later.

## 2.7 Covariance of Maxwell's equations

The covariance of electromagnetism is almost trivial to demonstrate with the machinery we have so far developed. We have seen that Maxwell's equations amount to the following two tensor equations,

$$\begin{aligned} &\partial_\mu F^{\mu\nu} = J^\nu \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{30}$$

Where we have noted that the definition above yields the homogenous Maxwell equations, provided we assume that eq. 2 relates components of the 4-potential $A_\mu$ and the $\vec{E}$ and

$\vec{B}$ fields. Now lets change coordinates from $x^\mu$ to $\bar{x}^\mu = \Lambda^\mu_\alpha x^\mu$. First, notice that because this transformation is linear it is it's own best linear approximation meaning $\Lambda^\mu_\alpha = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}$. Additionally, differentiating the inverse transformation $x^\mu = (\Lambda^{-1})^\mu_\alpha \bar{x}^\alpha$ yields $(\Lambda^{-1})^\mu_\alpha = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}$. Finally the chain rule is $\bar{\partial}_\mu = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\partial_\alpha$. Recalling the tensor transformation laws of eq. 9 we have (assuming the field tensor is a tensor as we prove in eq. 37)

$$\begin{aligned} &\bar{F}^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta} \\ &\bar{\partial}_\mu = (\Lambda^{-1})^\gamma_\mu \partial_\gamma \\ &\bar{J}^\nu = \Lambda^\nu_\alpha J^\alpha \end{aligned} \tag{31}$$

Now let us assemble the tensor Maxwell equations in the new coordinates

$$\bar{\partial}_\mu \bar{F}^{\mu\nu} = \bar{J}^\nu \tag{32}$$

So why does this equation hold ? Lets relate it to Maxwell's equations in the original coordinates as prescribed by eq. 31,

$$(\Lambda^{-1})^\gamma_\mu \partial_\gamma \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta} = \Lambda^\nu_\alpha J^\alpha \tag{33}$$

Thus,

$$(\Lambda^{-1})^\gamma_\mu \Lambda^\mu_\alpha \Lambda^\nu_\beta \partial_\gamma F^{\alpha\beta} = \Lambda^\nu_\alpha J^\alpha \tag{34}$$

Then since $(\Lambda^{-1})^\gamma_\mu \Lambda^\mu_\alpha = \delta^\gamma_\alpha$ we find,

$$\Lambda^\nu_\beta \partial_\gamma F^{\gamma\beta} = \Lambda^\nu_\alpha J^\alpha \tag{35}$$

Multiply this equation by the inverse of the matrix appearing above ($(\Lambda^{-1})^\rho_\nu$) and switch some indices to arrive at,

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{36}$$

All the steps above are reversible, therefore we have shown that the form of Maxwell's equations is the same in all inertial frames. That is to say that electromagnetism is a *relativistically covariant* theory as was claimed. Well, we should also show that the field tensor is defined in the new coordinates as it is in the original coordinates. That is almost immediate from it's definition in terms of the potential.

$$\begin{aligned} \bar{F}_{\mu\nu} &= \bar{\partial}_\mu \bar{A}_\nu - \bar{\partial}_\nu \bar{A}_\mu \\ &= (\Lambda^{-1})^\alpha_\mu \partial_\alpha (\Lambda^{-1})^\beta_\nu A_\beta - (\Lambda^{-1})^\beta_\nu \partial_\beta (\Lambda^{-1})^\alpha_\mu A_\alpha \\ &= (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &= (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu F_{\alpha\beta} \end{aligned} \tag{37}$$

This proves that the field tensor is a tensor. Calculations like those above can clearly be applied to any tensorial equation. This means if we want to write physics in a frame-independent manner we simply need to phrase the physics via tensors. Actually not all fundamental physics can be stated in terms of tensorial equations. There is another possible object used in physical theory, *spinors*, more on those later. To conclude, we find that electromagnetism combines naturally with special relativity. I think it is this consistency which made Special Relativity gain such wide acceptance. Special relativity goes hand in hand with electromagnetism and the validity of electromagnetism is well-established. Anyway, at least it is why I believe Special Relativity is a reasonable theory of physics. Experimentalists can disagree.

## 2.8 Coordinate free E&M

The nuts and bolts we have already covered, but it is nice to recast the thoughts of the last few sections in terms of differential forms. Unlike previous sections we will write the objects rather than just the components as is common in physics literature.

### 2.8.1 Homogeneous Maxwell's Equations are dF=0

For example the field strength is a $T_2^0(\mathbb{R}^4)$ tensor, thus in the $x^\mu$ coordinates,

$$F = F_{\mu\nu} dx^\mu \otimes dx^\nu \tag{38}$$

We know that F is an antisymmetric tensor thus it is also a differential form,

$$\begin{aligned} F &= \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2}\left(F_{0k} dx^0 \wedge dx^k + F_{ij} dx^i \wedge dx^j + F_{i0} dx^i \wedge dx^0\right) \\ &= \tfrac{1}{2}\left(2E_i dx^0 \wedge dx^i - \epsilon_{ijk} B_k dx^i \wedge dx^j\right) \\ &= E_i dx^0 \wedge dx^i - \tfrac{1}{2} B_i \epsilon_{ijk} dx^j \wedge dx^k \end{aligned} \tag{39}$$

Notice that $F_{\mu\nu}$ are the tensor components, that is why the $\frac{1}{2}$ appears in the equation above. But, beware the sums are taken over all values of the indices (not just increasing strings like we would have if we were working with a basis of forms). Consider then,

$$\begin{aligned} dF &= d(E_i dx^0 \wedge dx^i) - \tfrac{1}{2}\epsilon_{ijk} d(B_i dx^j \wedge dx^k) \\ &= \partial_\mu E_i dx^\mu \wedge dx^0 \wedge dx^i - \tfrac{1}{2}\epsilon_{ijk}\partial_\mu B_i dx^\mu \wedge dx^j \wedge dx^k \\ &= \partial_j E_k dx^j \wedge dx^0 \wedge dx^k - \tfrac{1}{2}\epsilon_{ijk}\partial_0 B_i dx^0 \wedge dx^j \wedge dx^k - \epsilon_{ijk}\tfrac{1}{2}\partial_l B_i dx^l \wedge dx^j \wedge dx^k \\ &= \tfrac{1}{2}\left(\partial_k E_j - \partial_j E_k - \epsilon_{ijk}\partial_0 B_i\right) dx^0 \wedge dx^j \wedge dx^k - \partial_i B_i dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{40}$$

In the last step we used that $dx^l \wedge dx^j \wedge dx^k = \epsilon^{ljk} dx^1 \wedge dx^2 \wedge dx^3$ and $\epsilon_{ijk}\epsilon^{ljk} = 2\delta_i^l$ to simplify the term on the right. If we require that $dF = 0$ we will recover the homogeneous Maxwell equations.

$$\begin{aligned} &\partial_k E_j - \partial_j E_k - \epsilon_{ijk}\partial_0 B_i = 0 \\ &\partial_i B_i = 0 \end{aligned} \tag{41}$$

Clearly the second equation is the no magnetic monopole equation $\nabla \cdot \vec{B} = 0$. The first equation looks a bit strange, multiply it by $\epsilon^{ljk}$ to make it look more familiar,

$$\begin{aligned} 0 &= \epsilon^{ljk}(\partial_k E_j - \partial_j E_k) - \epsilon^{ljk}\epsilon_{ijk}\partial_0 B_i \\ &= -2\epsilon^{ljk}\partial_j E_k - 2\delta_i^l \partial_0 B_i \\ &= 2(\epsilon_{ljk}\partial_j E_k + \partial_0 B_l) \\ &= 2(\nabla \times \vec{E} + \partial_t \vec{B})_l \end{aligned} \tag{42}$$

Next notice that the 4-potential is a 1-form,

$$A = A_\mu dx^\mu \tag{43}$$

If dF=0 (applying Maxwell's equations ) then by Poicaire's conjecture we know that there exists some 1-form $A$ such that $dA = F$. Such a 1-form can only be found locally in general, globally we'll need to supply multiple 1-forms to cover the whole space. More on these global issues later. Anyway, at least locally we can conclude,

$$\begin{aligned} dA &= d(A_\mu dx^\mu) \\ &= (\partial_\nu A_\mu) dx^\nu \wedge dx^\mu \\ &= \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \\ &= F_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= F \end{aligned} \tag{44}$$

Thus the homogeneous Maxwell's equation are compactly summarized as,

$$dF = d(dA) = d^2(A) = 0 \tag{45}$$

The tensor identity above is the "Bianchi identity", in components it can be rephrased as follows

$$\partial_\gamma F_{\mu\nu} + \partial_\mu F_{\nu\gamma} + \partial_\nu F_{\gamma\mu} = 0 \tag{46}$$

This can be derived from substituting $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ into the identity above,

$$\begin{aligned} \partial_\gamma F_{\mu\nu} &= \partial_\gamma \big(\partial_\mu A_\nu - \partial_\nu A_\mu\big) = \partial_\mu \partial_\gamma A_\nu - \partial_\nu \partial_\gamma A_\mu \\ \partial_\mu F_{\nu\gamma} &= \partial_\mu \big(\partial_\nu A_\gamma - \partial_\gamma A_\nu\big) = \partial_\mu \partial_\nu A_\gamma - \partial_\mu \partial_\gamma A_\nu \\ \partial_\nu F_{\gamma\mu} &= \partial_\nu \big(\partial_\gamma A_\mu - \partial_\mu A_\gamma\big) = \partial_\nu \partial_\gamma A_\mu - \partial_\mu \partial_\nu A_\gamma \end{aligned} \tag{47}$$

From the above it should be clear that the "Bianchi identity" is true, independent of the source content of the theory. As an undergraduate I tried to do this in components... I was not successful. Another way to get this identity is to take the exterior derivative then antisymmetrize as below,

$$\begin{aligned} dF &= (\partial_\gamma F_{\mu\nu}) dx^\gamma dx^\mu dx^\nu \\ &= \tfrac{1}{6}\left(\partial_\gamma F_{\mu\nu} + \partial_\mu F_{\nu\gamma} + \partial_\nu F_{\gamma\mu} - \partial_\nu F_{\mu\gamma} - \partial_\mu F_{\gamma\nu} - \partial_\gamma F_{\nu\mu}\right) dx^\gamma dx^\mu dx^\nu \\ &= \tfrac{1}{3}\left(\partial_\gamma F_{\mu\nu} + \partial_\mu F_{\nu\gamma} + \partial_\nu F_{\gamma\mu}\right) dx^\gamma dx^\mu dx^\nu \end{aligned} \tag{48}$$

Then as we know $dF = 0$ the "Bianchi" identity falls out naturally.

### 2.8.2 Inhomogeneous Maxwell's Equations are $d * F = - * j$

On Minkowski space we have a metric so the *Hodge dual* naturally induces forms which are dual to the forms above.

$$*F = \tfrac{1}{2} F^{\alpha\beta} \epsilon_{\alpha\beta\mu\nu} dx^\mu \wedge dx^\nu \tag{49}$$

In components this means,

$$*F_{\mu\nu} = \tfrac{1}{2}F^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu} = G_{\mu\nu} \tag{50}$$

Where we mention $G_{\mu\nu}$ to connect with the notation of Grffiths.

$$\begin{aligned}
*F_{\mu\nu}dx^\mu \wedge dx^\nu \quad &= \tfrac{1}{2}(F^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu})dx^\mu \wedge dx^\nu \\
&= \tfrac{1}{2}(F^{0\beta}\epsilon_{0\beta\mu\nu} + F^{i\beta}\epsilon_{i\beta\mu\nu})dx^\mu \wedge dx^\nu \\
&= \tfrac{1}{2}(F^{0j}\epsilon_{0j\mu\nu} + F^{i0}\epsilon_{i0\mu\nu} + F^{ij}\epsilon_{ij\mu\nu})dx^\mu \wedge dx^\nu \\
&= \tfrac{1}{2}(2F^{0j}\epsilon_{0jkl}dx^k \wedge dx^l + F^{ij}\epsilon_{0ijk}dx^0 \wedge dx^k) \\
&= \tfrac{1}{2}(-2E_j\epsilon_{jkl}dx^k \wedge dx^l - B_l\epsilon_{ijl}\epsilon_{ijk}dx^0 \wedge dx^k) \\
&= \tfrac{1}{2}(-2E_j\epsilon_{jkl}dx^k \wedge dx^l - 2B_k)dx^0 \wedge dx^k) \\
(mistake?) \qquad &= -E_j\epsilon_{jkl}dx^k \wedge dx^l - B_k dx^0 \wedge dx^k
\end{aligned} \tag{51}$$

Therefore, the dual to the field strength is obtained by the exchange of $\vec{B} \to \vec{E}$ and $\vec{E} \to -\vec{B}$ in the field strength,

$$(*F_{\mu\nu}) = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix} \tag{52}$$

Investigate the exterior derivative of the dual field strength,

$$\begin{aligned}
d*F \quad &= d(-\tfrac{1}{2}E_j\epsilon_{jkl}dx^k \wedge dx^l - B_k dx^0 \wedge dx^k) \\
&= -\tfrac{1}{2}\epsilon_{jkl}(\partial_\mu E_j)dx^\mu \wedge dx^k \wedge dx^l - (\partial_\mu B_k)dx^\mu \wedge dx^0 \wedge dx^k \\
&= -\tfrac{1}{2}\epsilon_{jkl}(\partial_0 E_j dx^0 + \partial_m E_j dx^m) \wedge dx^k \wedge dx^l - \partial_j B_k dx^j \wedge dx^0 \wedge dx^k \\
&= (-\tfrac{1}{2}\epsilon_{jkl}\partial_0 E_j + \partial_k B_l)dx^0 \wedge dx^k \wedge dx^l - \tfrac{1}{2}\epsilon_{jkl}\partial_m E_j dx^k \wedge dx^l \wedge dx^m \\
&= (-\tfrac{1}{2}\epsilon_{jlm}\partial_0 E_j + \partial_l B_m)dx^0 \wedge dx^l \wedge dx^m - (\partial_m E_m)dx^1 \wedge dx^2 \wedge dx^3
\end{aligned} \tag{53}$$

In the last step we used that $dx^k \wedge dx^l \wedge dx^m = \epsilon^{klm}dx^1 \wedge dx^2 \wedge dx^3$ and $\epsilon_{jkl}\epsilon^{klm} = 2\delta^m_j$ to simplify the term on the right. Comparing the above with eq. 28 we can clearly identify that these terms give part of the inhomogeneous Maxwell's equations. We just need to put the sources into a differential form, note the one form $j = j_\mu dx^\mu$ (where $(j^\mu) = (\rho, \vec{J})$) doesn't have the right degree. We need a 3-form and the *Hodge* dual to the current 1-form is a 3-form,

$$\begin{aligned}
*j \quad &= j^\alpha \epsilon_{\alpha\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho \\
&= (j^0\epsilon_{0\mu\nu\rho} + j^k\epsilon_{k\mu\nu\rho})dx^\mu \wedge dx^\nu \wedge dx^\rho \\
&= (j^0\epsilon_{0123}dx^1 \wedge dx^2 \wedge dx^3 + j^k\epsilon_{k0lm}dx^0 \wedge dx^l \wedge dx^m) \\
&= (\rho dx^1 \wedge dx^2 \wedge dx^3 - J_k\epsilon_{klm}dx^0 \wedge dx^l \wedge dx^m) \quad (mistake?)
\end{aligned} \tag{54}$$

Therefore we find the geometric statement of the inhomogeneous Maxwell's equations is,

$$d*F = -*j \tag{55}$$

Recall from our discussion of the Laplacian and forms that the coderivative raises the degree of the form by one and is related to the exterior derivative by $\delta = *d*$ (in Minkowski space). Therefore Maxwell's equations are

$$\delta F = -j \qquad dF = 0 \tag{56}$$

### 2.8.3 Invariants in action

As the *Hodge dual* creates a differential form we know that the dual field strength $*F$ is an antisymmetric tensor. Thus the following are invariant quantities,(using $G_{\mu\nu} = *F_{\mu\nu}$) and for two-forms in Minkowski space $** = -1$ (these formulae follow from the p-form metric from lecture. We proved that $(\alpha, \alpha) = \int \alpha \wedge *\alpha = \int \alpha_J \alpha^J dx^n$ , where J is a length-p multiindex and n is the manifold's dimension)

$$\begin{array}{l} F \wedge *F = (F^{\mu\nu} F_{\mu\nu}) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ *F \wedge *F = (-G^{\mu\nu} F_{\mu\nu}) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ F \wedge F = (F^{\mu\nu} G_{\mu\nu}) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{array} \tag{57}$$

Each of the above has physical significance, although the last two are the same upto sign which depends on the signature of the metric, it would be + in Euclidean space. Finally, note we can phrase the action using integration on forms.

$$S[A] = \int_{\mathbb{R}^4} (-\tfrac{1}{2} F \wedge *F - A \wedge *j) \tag{58}$$

In components this is the same as eq. 26. Explicitly,

$$\begin{array}{rl} S[A] & = \int_{\mathbb{R}^4} (-\frac{1}{2} F \wedge *F - A \wedge *j) \\ & = \int_{\mathbb{R}^4} (-\frac{1}{2}\frac{1}{2} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{array} \tag{59}$$

### 2.8.4 Gauge transformations in differential form notation

We discussed the idea of a gauge transformation in section 2.5. Now recast those thoughts in a coordinate free language. Note $A$ and $A' = A + d\lambda$ yield same field strength $F$,

$$F = dA' = d(A + d\lambda) = dA \tag{60}$$

This gauge freedom follows from $d^2$ being zero. So we see that there are many potentials which lead the same field strength. In components,

$$\begin{array}{ll} d\lambda & = (\partial_\mu \lambda) dx^\mu \\ & = \partial_t \lambda dt - \partial_x \lambda dx - \partial_y \lambda dy - \partial_z \lambda dz \\ (A'_\mu) & = (\phi + \partial_t \lambda, \vec{A} - \nabla \lambda) \end{array} \tag{61}$$

Thus adding $d\lambda$ amounts to adding $\partial_t \lambda$ to $\phi$ and subtracting $\nabla\lambda$ to $\vec{A}$, In order for the action to be gauge invariant we ought to have that $S[A] = S[A']$

$$\begin{array}{rl} S[A'] & = \int \left(-\frac{1}{2} d(A + d\lambda) \wedge *d(A + d\lambda) - (A + d\lambda) \wedge *j\right) \\ & = \int \left(-\frac{1}{2} dA \wedge *dA - A \wedge *j - d\lambda \wedge *j\right) \\ & = \int \left(-\frac{1}{2} F \wedge *F - A \wedge *j - d(\lambda \wedge *j) + \lambda \wedge d * j\right) \\ & = S[A] + \int (\lambda \wedge d * j) \end{array} \tag{62}$$

Where we assume that the integration region has $j = 0$ so we can throw away the boundary term above. Then gauge invariance gives that the above holds for arbitrary $\lambda$ and consequently $d * j = 0$. This is Noether's theorem at work, the symmetry of gauge invariance has lead us to a conservation law; here the conservation of charge,

$$\begin{aligned} d * j &= d(\rho dx^1 \wedge dx^2 \wedge dx^3 - \tfrac{1}{2} J_k \epsilon_{klm} dx^0 \wedge dx^l \wedge dx^m) \\ &= (\partial_t \rho) dt \wedge dx^1 \wedge dx^2 \wedge dx^3 - \tfrac{1}{2} \epsilon_{klm} (\partial_i J_k) dx^i \wedge dt \wedge dx^l \wedge dx^m) \\ &= \left(\partial_t \rho + \tfrac{1}{2} \epsilon_{klm} \epsilon^{ilm} \partial_i J_k\right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3) \\ &= (\partial_t \rho + \partial_i J_i)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= 0 \end{aligned} \tag{63}$$

Thus $\nabla \cdot \vec{J} = -\partial_t \rho$ which simply says that the current that diverges from some region must be the same as the loss of charge density. Here the conserved quantity is the charge, in general the terminology follows this even though the new conserved quantity is not charge. In gauge theories the "charge" is a general notion, but this is the fundamental case.

# 3 Magnetic Monopole

This example is very interesting as it necessitates the use of the Hopf bundle. Apparently the same time P.A.M. Dirac was working out the physical ramifications of magnetic charge, a mathematician Heinz Hopf was working out what we now call Hopf bundles. We will examine how these two studies merge, this was found explicitly by Wu and Yang in the mid-1970's. Interestingly it can be shown that topological considerations suggest the magnetic charge is quantized if it exists. Additionally, the coupling of the magnetic monopole to some electrically charged wave function indicates that electric charge is quantized. This is an interesting proof as we have no other proof for this observed fact of nature.

## 3.1 The fiber bundlelation of E&M

As we have discussed in detail in lecture, we may identify the **g**-valued 1-form $A$ with the pullback of a connection on a principal fiber bundle which has Lie group G with Lie algebra **g**. Since the components of the field strength and potential are real-valued functions of spacetime it's fairly obvious that our gauge group's Lie algebra is $\mathbb{R}$. Thus we are led to make $G = U(1)$. Another motivation comes later on when we consider scalar matter fields, the wavefunctions are complex functions and the phase of the wavefunction admits $U(1)$ rotations.

Weyl derived electromagnetism by postulating the existence of a "local" $U(1)$ symmetry. He showed that by introducing such a symmetry one is naturally led to introduce a gauge field. Originally in 1919 he tried to do this as an "external" symmetry. His original attempt was plagued with many strange predictions, imaginary mass for example. It was a decade later, after the work of Fock and London made clearer the interaction of the wavefunction's phase and $E\&M$, that Weyl reinterpreted his original attempt and made the $U(1)$ an "internal" symmetry. It was doubtless his work that led Yang and others to what we know as gauge theory today. (See O'raierfertaigh's *Dawn of Gauge Theory* for the history and chapter 3 of Ryder's QFT for the derivation of Maxwell's equations from a local $U(1)$ symmetry). Nowadays we know that a "local" $U(1)$ symmetry can be endowed to a set of matter fields by defining them to be sections of the associated fiber bundle of a PFB with $G = U(1)$. Before we study the associated bundle we ought to study the relevant PFB which describes the gauge field itself. Consider then a PFB with $G = U(1)$ over some region of 4-dimensional Minkowski spacetime,

$$\begin{aligned} &\mathcal{D} = d \\ &\mathcal{A} = \tfrac{g}{i} A \\ &\mathcal{F} = \tfrac{g}{i} F = \mathcal{D}\mathcal{A} \\ &\mathcal{A} \longrightarrow \mathcal{A} + d\Lambda \\ &\mathcal{F} \longrightarrow \mathcal{F} \end{aligned} \tag{64}$$

Where $\Lambda = \frac{m}{i}\lambda$ in comparison to earlier discussion. The topology of the bundle depends on the configuration of the sources. Upto now we have assumed that only *electric* charge and current exists. This is an experimental fact upto now, so this lack of generalization is probably ok. In a typical E&M course one learns how to find solutions for a variety of sources,but one common thread runs thru the examples. Modulo the location of the sources the potentials *can* be specified globally. It is in fact necessary to "fix the gauge" before finding an explicit solution ( popular gauges are the Coulomb gauge which sets $\nabla \cdot \vec{A} = 0$, or the Lorentz gauge which sets $\partial_\mu A^\mu = 0$ ). It is similar to the case in classical mechanics where before giving an explicit potential energy formula you must first pick what should be the zero energy. Once we choose a gauge to work out the solutions we will find single valued solutions for the scalar and vector potentials. It maybe they cannot be defined globally, but that should come from some quirk in the charge configuration, those locations we will refer to as *physical singularities.* If we found after fixing the gauge that there were points with no source charges that had the potentials ill-defined we will call those points *unphysical singularities* of the potential.

To summarize my claim, classical magnetic charge-free E&M has only *physical singularities.* What this means is that the fiber-bundle formulation is somewhat superfluous, we have no need of the transition functions. Those regions of spacetime which are connected possess global potentials. The boundaries to those regions are formed by genuine physical obstructions, the source charges and currents. If you would like to read a careful argument for the uniqueness of the potential in electrostatics I recommend pages 116-121 of Griffiths. We will see next that this is a fortunate turn of events, once magnetic charge enters we cannot generally find potentials free of *unphysical singularities.* In the case of a single magnetic monopole the *unphysical singularities* are the Dirac Strings.

## 3.2 Introduction to symmetric Electromagnetism

I'm not sure what the official name of this beast is, but it is a natural extension of conventional electromagnetism. It is a homework problem in Griffith's (12.57) to write the dualized Maxwell's equations. If you look at Maxwell's equations there is clearly an asymmetry between the equations. We can remedy that by suggesting that there is magnetic charge (with density $\rho_m$) and magnetic current (with density $\vec{J}_m$) so that Maxwell's equations take the symmetric form:

$$\begin{aligned} &\nabla \cdot \vec{E} = \rho_e &\qquad &\nabla \cdot \vec{B} = \rho_m \\ &\nabla \times \vec{E} + \partial_t \vec{B} = -\vec{J}_m &\qquad &\nabla \times \vec{B} - \partial_t \vec{E} = \vec{J}_e \end{aligned} \tag{65}$$

I adorned the electric currents with an "e" just to be fair. At the present I don't understand how to write these equations in terms of a single natural PFB. It seems to me that if there is both electric and magnetic charge then we cannot have that $dF = 0$ which means we can't write $F = dA$, so the Field strength can't be identified as the curvature of some connection. However, there must be some caveat to my concern because people do study systems with both types of charges. Hopefully we'll resolve this as we go on.

Before we return to this rather general question lets study in depth the properties of the fields and potentials of a single static (unmoving always existing) magnetic charge. We place the monopole (single charge) at the origin. Then the magnetic monopole with magnetic charge M (this is g in other treatments) should have field $\vec{B} = \frac{M\vec{r}}{r^3}$, in close analogy to the Coulomb field of a point electric charge.

Notice that this $\vec{B}$ field can be found by applying the magnetic version of Gauss' law. A point magnetic charge of M has magnetic charge density $\rho_m = 4\pi M\delta(\vec{r})$. Then naturally the "no magnetic monopoles equation" is modified so $\vec{B}$ has a divergence,

$$\nabla \cdot \vec{B} = 4\pi M\delta(\vec{r}) \tag{66}$$

Integration over any gaussian sphere (S) or radius r centered at the origin yields for the l.h.s of the above,

$$\int_S \nabla \cdot \vec{B} d\tau = \oint_{\partial S} B \cdot d\vec{A} = 4\pi r^2 B \tag{67}$$

The last step followed from the spherical symmetry of the monopole density. While integration of the r.h.s gives,

$$\int_S 4\pi M\delta(\vec{r}) d\tau = 4\pi M \int_S \delta(\vec{r}) d\tau = 4\pi M \tag{68}$$

Thus we find that $4\pi r^2 B = 4\pi M$ thus $B = \frac{M}{r^2}$.

### 3.3 Physics approach to the Magnetic Monopole: Dirac Strings

#### 3.3.1 Monopole potential in Cartesian coordinates

We use standard coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ and spherical coordinates $(r, \theta, \phi)$ with radius $r$ given by $r^2 = x^2 + y^2 + z^2$, polar angle $0 \leq \theta \leq \pi$, and azimuthial angle $0 \leq \phi \leq 2\pi$. We propose that the following potential describes a magnetic monopole,

$$\mathcal{A} = \tfrac{im}{2}\tfrac{1}{r(z-r)}\big(xdy - ydx\big) \tag{69}$$

To prove this potential is indeed the potential of a magnetic monopole we take the covariant derivative of the potential to find the field strength. Here the group is U(1) so the covariant derivative is just the exterior derivative,

$$\begin{aligned} d\mathcal{A} &= d\big(\ \tfrac{im}{2}\tfrac{1}{r(z-r)}(xdy - ydx)\ \big) \\ &= \tfrac{im}{2}\big(\ d(\tfrac{x}{r(z-r)})dy - d(\tfrac{y}{r(z-r)}dx)\ \big) \\ &= \tfrac{im}{2}\big(d(\alpha dy) - d(\beta dx)\big) \\ &= \tfrac{im}{2}\big((\partial_x\alpha)dx \wedge dy + (\partial_z\alpha)dz \wedge dy - (\partial_y\beta)dy \wedge dx - (\partial_z\beta)dz \wedge dx\big) \\ &= \tfrac{im}{2}\big((\partial_x\alpha + \partial_y\beta)dx \wedge dy - (\partial_z\alpha)dy \wedge dz - (\partial_z\beta)dz \wedge dx\big) \end{aligned} \tag{70}$$

The definition of the abbreviations $\alpha$ and $\beta$ should be manifest from the expression above. Also note we used the skew-property of $\wedge$ to combine two terms. The radial coordinate can be expressed as $r = \sqrt{x^j x^j}$, where the sum is implied and goes over $j = 1, 2, 3$. Consider then,

$$\tfrac{\partial r}{\partial x^k} = \tfrac{1}{2r}\tfrac{\partial}{\partial x^k}\big[x^j x^j\big] = \tfrac{x^k}{r} \tag{71}$$

This result holds for $k = 1, 2, 3$ or $x, y, z$ if you please. We'll use it throughout the calculations below.

$$\begin{aligned} \partial_x\alpha &= \tfrac{\partial}{\partial x}\big[\tfrac{x}{r(z-r)}\big] \\ &= \tfrac{1}{r^2(z-r)^2}\big[r(z-r)\tfrac{\partial}{\partial x}(x) - x\tfrac{\partial}{\partial x}(r(z-r))\big] \\ &= \tfrac{1}{r^2(z-r)^2}\big[r(z-r) - x\big(\ (\tfrac{\partial}{\partial x}r)(z-r) + r\tfrac{\partial}{\partial x}(z-r)\ \big)\big] \\ &= \tfrac{1}{r^2(z-r)^2}\big[r(z-r) - x\big(\ \tfrac{x}{r}(z-r) - r\tfrac{x}{r}\ \big)\big] \\ &= \tfrac{1}{r^2(z-r)^2}\big[r(z-r) - \tfrac{x^2}{r}(z-r) + r\tfrac{x^2}{r}\ \big)\big] \\ &= \tfrac{1}{r^3(z-r)^2}\big[r^2(z-r) - x^2(z-r) + rx^2\big] \\ &= \tfrac{1}{r^3(z-r)^2}\big[(z-r)[r^2 - x^2] + rx^2\big] \end{aligned} \tag{72}$$

Likewise we can calculate,

$$\partial_y\beta = \tfrac{1}{r^3(z-r)^2}\big[(z-r)[r^2 - y^2] + ry^2\big] \tag{73}$$

Now something neat happens when we add these together, remember $r^2 = x^2 + y^2 + z^2$,

$$\begin{aligned} \partial_x\alpha + \partial_y\beta &= \tfrac{1}{r^3(z-r)^2}\big[(z-r)[2r^2 - x^2 - y^2] + r(x^2 + y^2)\big] \\ &= \tfrac{1}{r^3(z-r)^2}\big[(z-r)[r^2 + z^2] + r(r^2 - z^2)\big] \\ &= \tfrac{1}{r^3(z-r)^2}\big[zr^2 - rz^2 + z^3 - r^3 + r^3 - rz^2\big] \\ &= \tfrac{1}{r^3(z^2-2rz+r^2)}\big[z\big(z^2 - 2rz + r^2\big)\big] \\ &= \tfrac{z}{r^3} \end{aligned} \tag{74}$$

Moving on, calculate

$$\begin{aligned}
\partial_z\alpha &= \tfrac{\partial}{\partial z}\left[\tfrac{x}{r(z-r)}\right] \\
&= \tfrac{x}{r^2(z-r)^2}\left[-\tfrac{\partial}{\partial z}(r(z-r))\right] \\
&= -\tfrac{x}{r^2(z-r)^2}\left[(\tfrac{\partial}{\partial z}r)(z-r)+r(1-\tfrac{\partial}{\partial z}r)\right] \\
&= -\tfrac{x}{r^2(z-r)^2}\left[\tfrac{z}{r}(z-r)+r-r\tfrac{z}{r}\right] \\
&= -\tfrac{x}{r^3(z-r)^2}\left[z(z-r)+r^2-rz\right] \\
&= -\tfrac{x}{r^3(z^2-2rz+r^2)}\left[z^2-2rz+r^2\right] \\
&= -\tfrac{x}{r^3}
\end{aligned} \tag{75}$$

Replace $x \to y$ in the above and the calculation follows similarly,

$$\partial_z\beta = -\tfrac{y}{r^3} \tag{76}$$

Let assemble our results,

$$d\mathcal{A} = \tfrac{im}{2r^3}\big(zdx\wedge dy + xdy\wedge dz + ydz\wedge dx\big) \tag{77}$$

Lets switch back to the $(x^0, x^1, x^2, x^3)$ notation, we can read off the components of the field strength directly from the above. Just remember that $d\mathcal{A} = \frac{g}{i}F$ and $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ where in particular $F_{12} = B_3$, $F_{31} = B_2$, $F_{23} = B_1$. Thus,

$$\begin{aligned}
B_x &= \tfrac{-m}{2g}\tfrac{x}{r^3} \\
B_y &= \tfrac{-m}{2g}\tfrac{y}{r^3} \\
B_z &= \tfrac{-m}{2g}\tfrac{z}{r^3}
\end{aligned} \tag{78}$$

Which is the desired result, we identify that the magnetic charge $M = \frac{-m}{2g}$. We note that there is something very odd about this solution, it is clear that the potential is singular along the positive z-axis $z = r$, yet there is only charge at the origin. This strange singularity is the "Dirac String", I would say it is completely unphysical except at the origin. At the origin there is charge so we expect the potential to be singular there.

### 3.3.2 Monopole potential in spherical coordinates

We endeavor to change coordinates on the potential discussed in last section. Recall that

$$\begin{aligned}
x^1 &= r\cos\phi\sin\theta \\
x^2 &= r\sin\phi\sin\theta \\
x^3 &= r\cos\theta
\end{aligned} \tag{79}$$

Remember that $dx^i = \frac{\partial x^i}{\partial r}dr + \frac{\partial x^i}{\partial \phi}d\phi + \frac{\partial x^i}{\partial \theta}d\theta$ thus,

$$\begin{aligned}
dx^1 &= \cos\phi\sin\theta dr - r\sin\phi\sin\theta d\phi + r\cos\phi\cos\theta d\theta \\
dx^2 &= \sin\phi\sin\theta dr + r\cos\phi\sin\theta d\phi + r\sin\phi\cos\theta d\theta \\
dx^3 &= \cos\theta dr - r\sin\theta d\theta
\end{aligned} \tag{80}$$

Its a simple matter to change coordinates with the above worked out. Substituting sphericals into eq.69,Notice that,

$$\begin{aligned} x^1dx^2 &= r\cos\phi\sin\theta\big[\sin\phi\sin\theta dr + r\cos\phi\sin\theta d\phi + r\sin\phi\cos\theta d\theta\big] \\ x^2dx^1 &= r\sin\phi\sin\theta\big[\cos\phi\sin\theta dr - r\sin\phi\sin\theta d\phi + r\cos\phi\cos\theta d\theta\big] \end{aligned} \tag{81}$$

Or, distributing

$$\begin{aligned} x^1dx^2 &= r\cos\phi\sin\phi(\sin\theta)^2dr + r^2(\cos\phi)^2(\sin\theta)^2d\phi + r^2\cos\phi\sin\theta\sin\phi\cos\theta d\theta \\ x^2dx^1 &= r\sin\phi\sin\phi(\sin\theta)^2dr - r^2(\sin\phi)^2(\sin\theta)^2d\phi + r^2\cos\phi\sin\theta\sin\phi\cos\theta d\theta \end{aligned} \tag{82}$$

Clearly only the middle term survives when we compute the difference,

$$\begin{aligned} x^1dx^2 - x^2dx^1 &= r^2\big[(\cos\phi)^2(\sin\theta)^2 + (\sin\phi)^2(\sin\theta)^2\big]d\phi \\ &= r^2(\sin\theta)^2d\phi \\ &= r^2(1+\cos\theta)(1-\cos\theta)d\phi \end{aligned} \tag{83}$$

Thus in total we find,

$$\begin{aligned} \mathcal{A} &= \tfrac{im}{2}\tfrac{1}{r(z-r)}\big(xdy - ydx\big) \\ &= \tfrac{im}{2}\tfrac{1}{r^2(\cos\theta-1)}r^2(1+\cos\theta)(1-\cos\theta)d\phi \\ &= -\tfrac{im}{2}(1+\cos\theta)d\phi \end{aligned} \tag{84}$$

We've hidden the singularity ( at $\theta = 0$ ) by dividing by zero. That's ok though because we'll be primarily interested in studying how this connection behaves at the equator $(\theta = \frac{\pi}{2})$. I remarked earlier that the Dirac String along $z = r$ is artificial. One aspect of that comment is that we could easily write down another potential very similar to the one considered here by putting $z + r$ in the place of $z - r$. The calculation would go almost the same, and at the end instead of $z - r$ cancelling the $(1 - \cos\theta)$ term we would see $z + r$ cancel the $(1 + \cos\theta)$ term. This means if we use two locally defined potentials we can cover the whole sphere. $\mathcal{A}_S$ for the southern hemisphere and $\mathcal{A}_N$ for the northern hemisphere,

$$\begin{aligned} \mathcal{A}_S &= -\tfrac{im}{2}(1+\cos\theta)d\phi \qquad \theta \neq 0 \\ \mathcal{A}_N &= \tfrac{im}{2}(1-\cos\theta)d\phi \qquad \theta \neq \pi \end{aligned} \tag{85}$$

These disagree at the equator $(\theta = \frac{\pi}{2})$, in particular $\mathcal{A}_N = -\mathcal{A}_S = \frac{im}{2}d\phi$. Both of these potentials describe the same $\vec{B}$ field, therefore they must differ by a gauge transformation. On the equator,

$$\mathcal{A}_N - \mathcal{A}_S = imd\phi \tag{86}$$

This may not appear to troubling to you, but consider that this gauge transformation moves the singularity from the Northern to the Southern pole.

$$\mathcal{A}_N = \mathcal{A}_S + imd\phi \tag{87}$$

This looks like a standard gauge transformation but $\phi$ is multiply valued at the poles so technically this is a singular gauge transformation. Clearly one would like to avoid such if possible. The solution is almost silly, we just view these as the local formulae for a connection and thus we only need discuss the gauge transformation on the intersection of their domains. The formulae remain the same, but we have a careful prescription for how the local formulas should paste together. Namely the PFB machinery.

### 3.4 Principal Fiber Bundles (PFB)

We now quickly review the quintessential features of a PFB. A PFB consists of the following ingredients,

P= bundle space

M= base space

$\{U_\alpha\}$ covering M

$\pi : P \longrightarrow M$ projection map

G= group= fiber $= \pi^{-1}(x)$ for x in M

$\Psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow \mathcal{U}_\alpha \times G$ trivializing maps

We assume that P, M and G are manifolds and that $\pi$ is smooth with respect to the manifold structure. The requirement that the trivializing maps are diffeomorphisms yield the qualitative statement that the bundle is "locally a cartesian product". In addition to the structure indicated so far, we also need some further statements about how the Lie group G interacts with these objects.

First there must exist a group action $\sigma : P \times G \longrightarrow P$ denoted by $u \cdot g = \sigma(u, g)$ for $u \in P$ and $g \in G$. This mapping must be smooth and satisfy for $g, h, e \in G$ and $u \in P$

$$\begin{array}{lll} i.) & u \cdot (gh) & = (u \cdot g) \cdot h \\ ii.) & u \cdot e & = u \\ iii.) & u \cdot g & = u \implies g = e \end{array} \tag{88}$$

Where "e" is the group identity. An action satisfying condition iii. is called free. Second the group must move along the fibers in P, meaning for $u \cdot G = \{u \cdot g | g \in G\}$

$$\pi^{-1}(\pi(u)) \;\; = u \cdot G \tag{89}$$

Finally the group must satisfy, for all $u \in \pi^{-1}(U_\alpha)$,

$$\Psi_\alpha(u \cdot g) \;\; = \Psi_\alpha(u) \cdot g \tag{90}$$

Where we have defined $(x, h) \cdot g = (x, h \cdot g)$.

### 3.5 Principal Fiber Bundle with $M = S^2$ and $G = U(1)$

This section is basically example is 9.11 in Nakahara. We take the base space to be $S^2$ which will suffice for a static spherically symmetric system. The standard sphere $S^2$ is the set of all points in $\mathbb{R}^3$ such that $x^2 + y^2 + z^2 = 1$. Two charts will cover the sphere,

$$\begin{array}{ll} U_1 & = S^2 - \{(0, 0, -1)\} \\ U_2 & = S^2 - \{(0, 0, 1)\} \end{array} \tag{91}$$

This choice of charts follows Gockeler and Schucker. Now we consider what it means to say we have a PFB over $S^2$ with group $U(1)$. Notice that I'm leaving P quite arbitrary, in fact we are discussing a number of bundles at once. To see this lets study how the group acts on the equator which is homotopic to the intersection of the charts. The equator is $z = 0$, or in Nakahara we would say $\xi_3 = 0$. Study then a point u in P that resides in some fiber over the equator.

$$(\pi(u), \exp(i\alpha_1)) \quad = (\pi(u), t_{12}\exp(i\alpha_1)) \tag{92}$$

Let the transition function $t_{12} = \exp(im\phi)$ then we see that for this function to be single-valued on the equator we need that m is in $\mathbb{Z}$. Each choice of m will give us a topologically distinct PFB. For example,

$$\begin{array}{lll} m = 0 \implies P = S^2 \times U(1) & \textit{trivial bundle} \\ m = 1 \implies P = S^3 & \textit{Hopf bundle} \end{array} \tag{93}$$

One can also find the case m=2 occurring as an asymptotic limit of the t'Hooft - Polyakov monopole. We can see the similarity between the transition functions of this PFB and the gauge transformations of the Dirac monopole. To make this explicit we should define sections and connections, I leave that to the notes for now...

## 3.6 Hopf Bundle

This is the case m=1, essentially it is example 9.12 from Nakahara, but I have borrowed many nice results from Gockeler and Schucker and adopted much notation from pgs 140-141, 164-166, 172-174. The bundle space is

$$S^3 = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 \mid (y_1)^2 + (y_2)^2 + (y_3)^2 + (y_4)^2 = 1\} \tag{94}$$

It is convenient to use complex notation,

$$\begin{array}{l} z_1 = y_1 + iy_2 \\ z_2 = y_3 + iy_4 \\ S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \end{array} \tag{95}$$

We define the Hopf map $\pi : S^3 \longrightarrow S^2$ as follows,

$$\pi(z_1, z_2) = (\ \bar{z}_1 z_2 + \bar{z}_2 z_1\ ,\ i(\bar{z}_2 z_1 - \bar{z}_1 z_2)\ ,\ |z_1|^2 - |z_2|^2\ ) \tag{96}$$

This means at the level of real notation that,

$$\begin{array}{l} \pi(z_1, z_2) = (x, y, z) \\ x = 2(y_1 y_3 + y_2 y_4) \\ y = 2(y_2 y_3 - y_1 y_4) \\ z = (y_1)^2 + (y_2)^2 - (y_3)^2 - (y_4)^2 \end{array} \tag{97}$$

A short calculation will show this maps into $S^2$, but the surjectivity is more subtle, we'll find an indirect proof of it a little later.

### 3.6.1 Charts on the Hopf Bundle

Let us make the chart structure on $S^2$ explicit. We follow the nonstandard stereographic projections given in Nakahara. They're non-standard because both hemisphere's map to the interior of the disk bounded by the equator. We map $U_S$ to the equatorial disk (which has coordinates $(X, Y)$, I capitalized them to avoid confusion with x,y previously used for $S^2$ coordinates) by the stereographic projection,

$$(X, Y) \quad = (\tfrac{x}{1-z}, \tfrac{y}{1-z}) \tag{98}$$

This is smooth since $U_S$ does not contain the north pole $(0, 0, 1)$ where this map is singular. Then identify $(X, Y)$ with $Z = X + iY$ to find

$$\begin{aligned} Z &= X + iY \\ &= \tfrac{x+iy}{1-z} \\ &= \tfrac{2(y_1y_3+y_2y_4)+2i(y_2y_3-y_1y_4)}{(y_1)^2+(y_2)^2+(y_3)^2+(y_4)^2-((y_1)^2+(y_2)^2-(y_3)^2-(y_4)^2)} \\ &= \tfrac{2(y_1+iy_2)(y_3-iy_4)}{2(y_3+iy_4)(y_3-iy_4)} \\ &= \tfrac{y_1+iy_2}{y_3+iy_4} \\ &= \tfrac{z_1}{z_2} \end{aligned} \tag{99}$$

Similarly we define a chart on $U_N$, map to $(U, V)$ on the equatorial plane

$$(U, V) \quad = (\tfrac{x}{1+z}, \tfrac{-y}{1+z}) \tag{100}$$

This is smooth since $U_N$ does not contain the south pole $(0, 0, -1)$ where this map is singular. Then identify $(U, V)$ with $W = U + iV$ to find by algebra just like the above,

$$W \quad = U + iV \quad = \tfrac{z_2}{z_1} \tag{101}$$

So we can relate the coordinate patches on $S^2$ according to the simple formula,

$$Z = \tfrac{1}{W} \tag{102}$$

This holds on the intersection of the charts, in particular the equator. Define the following trivializations (changing Nakahara's $z_0 \to z_1$ and $z_1 \to z_2$ ),

$$\begin{aligned} \Psi_S(z_1, z_2) &= (Z, \tfrac{z_1}{|z_1|}) = (\tfrac{z_1}{z_2}, \tfrac{z_1}{|z_1|}) \quad for\ z_i \in U_S \\ \Psi_N(z_1, z_2) &= (W, \tfrac{z_2}{|z_2|}) = (\tfrac{z_2}{z_1}, \tfrac{z_2}{|z_2|}) \quad for\ z_i \in U_N \end{aligned} \tag{103}$$

It should be clear that $\Psi_S : \pi^{-1}(U_S) \longrightarrow U_S \times U(1)$ and $\Psi_N : \pi^{-1}(U_N) \longrightarrow U_N \times U(1)$. Let W be in $\pi^{-1}(z = 0)$ that is take a point in the bundle space $S^3$ over the equator ( where $Z = \frac{1}{W}$ and $|z_1| = \frac{1}{\sqrt{2}} = |z_2|$ following from $z = 0$ ) and study,

$$\begin{aligned} (\Psi_S \circ \Psi_N^{-1})(W, \sqrt{2}z_2) &= \Psi_S(z_1, z_2) \\ &= (Z, \tfrac{z_1}{|z_1|}) \\ &= (\tfrac{1}{W}, \sqrt{2}z_1) \\ &= (\tfrac{1}{W}, t_{SN}\sqrt{2}z_2) \qquad definition\ of\ t_{SN} \end{aligned} \tag{104}$$

We should have that $t_{SN} : U(1) \to U(1)$, and we do,

$$\sqrt{2}z_1 = t_{SN}\sqrt{2}z_2 \quad \Longrightarrow \ t_{SN} = \tfrac{z_1}{z_2} = x - iy \in U(1) \tag{105}$$

Noting that $|t_{SN}|^2 = x^2 + y^2 = 1$ because z=0 on the equator. Similarly we can show, $t_{NS} = \frac{z_2}{z_1}$. All together then we can verify that these transition functions do satisfy the requirements they ought,

$$t_{NS}t_{SN} = \tfrac{z_2}{z_1}\tfrac{z_1}{z_2} = 1 \quad \Longrightarrow \ {t_{NS}}^{-1} = t_{SN} \tag{106}$$

#### 3.6.2 The U(1) action on the Hopf Bundle

We need to write a group action $\sigma : S^3 \times U(1) \longrightarrow S^3$ denoted by $u \cdot g = \sigma(u, g)$ for $u \in S^3$ and $g \in U(1)$. For this bundle define, for each $g \in U(1)$ and $(z_1, z_2) \in S^3$

$$(z_1, z_2) \cdot g = (gz_1, gz_2) \tag{107}$$

This is an action because,

$$\begin{aligned} &i.) \ \ (z_1, z_2) \cdot gh &&= (ghz_1, ghz_2) \\ & &&= (hz_1, hz_2) \cdot g \\ & &&= ((z_1, z_2) \cdot h) \cdot g \\ \\ &ii.) \ \ (z_1, z_2) \cdot 1 &&= (1z_1, 1z_2) \\ & &&= (z_1, z_2) \\ \\ &iii.) \ \ (z_1, z_2) \cdot g &&= (z_1, z_2) \implies g = 1 \end{aligned} \tag{108}$$

Not quite so easily we show $u \cdot U(1) = \pi^{-1}(\pi(u))$, where our $u = (z_1, z_2)$ and $\pi((z_1, z_2)) = (x, y, z)$. Consider then,

$$\pi( \ (z_1, z_2) \cdot g \ ) \ \ = \pi( \ (gz_1, gz_2) \ ) = (\bar{x}, \bar{y}, \bar{z}) \tag{109}$$

Where we endeavor to find $\bar{x}$, $\bar{y}$, and $\bar{z}$ now

$$\begin{aligned} \bar{z} \ &= |gz_1|^2 - |gz_2|^2 \\ &= |g|^2(|z_1|^2 - |z_2|^2) \\ &= (|z_1|^2 - |z_2|^2) = z \end{aligned} \tag{110}$$

Next consider,

$$2(gz_i)(g\bar{z}_j) = 2|g|^2 z_i\bar{z}_j = z_i\bar{z}_j \tag{111}$$

it quickly follows from this that $\bar{x} = x$ and $\bar{y} = y$. Thus we have

$$\pi( \ (z_1, z_2) \cdot U(1) \ ) \ \ = \pi((z_1, z_2)) \tag{112}$$

which shows that $u \cdot U(1) = \pi^{-1}(\pi(u))$ for $u \in S^3$. Finally, we need to make sure we can pull the group action out of the trivializing maps. That is check that $\Psi_S((z_1, z_2) \cdot g) = \Psi_S(z_1, z_2) \cdot g$.

$$\begin{aligned} \Psi_S((z_1, z_2) \cdot g) &= (\Psi_S(gz_1, gz_2)) \\ &= (\tfrac{gz_1}{gz_2}, \tfrac{gz_2}{|gz_2|}) \\ &= (\tfrac{z_1}{z_2}, \tfrac{gz_2}{|z_2|}) \\ &= (\tfrac{z_1}{z_2}, \tfrac{z_2}{|z_2|}) \cdot g \\ &= \Psi_S(z_1, z_2) \cdot g \end{aligned} \tag{113}$$

Where we have assumed the group acts on the trivializing space in the usual manner, $(\vec{x}, g) \cdot h = (\vec{x}, gh)$. Clearly the proof for $\Psi_N$ will be very similar.

### 3.6.3 Hopf meets Dirac

We can see that the transition function that arises in the context of the gauge transformations for the magnetic monopole was $\exp{(im\phi)}$, just consider the formula $\mathcal{A}_N = \mathcal{A}_S + imd\phi$ and remember that we proved that $\mathcal{A}_N = \mathcal{A}_S + t_{NS}{}^{-1} dt_{NS}$. This is of course assuming that the mathematics of the monopole truly follows the PFB set-up, once we make that connection we can verify that the Hopf Bundle is the magnetic monopole with charge $m = 1$. Lets connect the dots,

$$\begin{aligned} t_{NS} &= \tfrac{z_2}{z_1} \\ &= x + iy \\ &= r\cos\phi\sin\theta + ir\sin\phi\sin\theta \\ &= r\sin\theta(\cos\phi + i\sin\phi) \\ &= r\sin\theta\exp{(i\phi)} \\ &= \exp{(i\phi)} \end{aligned} \tag{114}$$

We have used that $\theta = \frac{\pi}{2}$ at the equator, and $r = 1$ because its the standard $S^2$. Thus the bundle of the m=1 Dirac monopole is the Hopf bundle, they share the same base-space and transition functions.

### 3.6.4 Local Sections of Hopf

We introduce an interesting parametrization of $S^3$ for the convenience of this section.

$$S^3 = \{(\cos{(\tfrac{1}{2}\theta)}\exp{(i\psi_1)}, \sin{(\tfrac{1}{2}\theta)}\exp{(i\psi_2)} \mid 0 \leq \theta \leq \pi,\ \psi_1, \psi_2 \in \mathbb{R}\} \tag{115}$$

This parametrization replaces $z_1$ and $z_2$ as follows,

$$\begin{aligned} z_1 &= \cos{(\tfrac{1}{2}\theta)}\exp{(i\psi_1)} \\ z_2 &= \sin{(\tfrac{1}{2}\theta)}\exp{(i\psi_2)} \end{aligned} \tag{116}$$

Its not hard to see that $|z_1|^2 + |z_2|^2 = 1$ upon substituting the new parametrization, so it really is a parametrization of $S^3$. We could be more careful with the domains of the new parameters, as it stands they are multiply valued. Anyway, lets see how the Hopf map behaves in the new coordinates,

$$\begin{aligned}\pi(z_1,z_2) &= (\ \bar{z}_1 z_2 + \bar{z}_2 z_1\ ,\ i(\bar{z}_2 z_1 - \bar{z}_1 z_2)\ ,\ |z_1|^2 - |z_2|^2\ ) \\ &= (\sin(\theta)\tfrac{1}{2}(e^{i(\psi_2-\psi_1)} + e^{-i(\psi_2-\psi_1)}), \sin(\theta)\tfrac{1}{2i}(e^{i(\psi_2-\psi_1)} - e^{-i(\psi_2-\psi_1)}), \cos(\theta)) \\ &= (\sin(\theta)\cos(\psi_2-\psi_1),\ \sin(\theta)\sin(\psi_2-\psi_1),\ \cos(\theta))\end{aligned}$$

Its not hard to see that $\pi$ maps into $S^2$ in these coordinates. We now define local sections $\sigma_i : U_i \longrightarrow \pi^{-1}(U_i)$ by,

$$\begin{aligned}\sigma_N(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta) &= (\ \cos(\tfrac{1}{2}\theta),\ e^{i\phi}\sin(\tfrac{1}{2}\theta)\ ) \qquad for\ U_N \\ \sigma_S(\cos\phi\sin\theta,\ \sin\phi\sin\theta,\cos\theta) &= (\ e^{-i\phi}\cos(\tfrac{1}{2}\theta),\sin(\tfrac{1}{2}\theta)\ ) \qquad for\ U_S\end{aligned} \tag{117}$$

We should verify that these are local sections, that is show $\pi\circ\sigma_i = id_{U_i}$

$$\begin{aligned}\pi(\sigma_N(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)) &= \pi(\ \cos(\tfrac{1}{2}\theta),\ e^{i\phi}\sin(\tfrac{1}{2}\theta)\ ) \\ &= \pi(\ \cos(\tfrac{1}{2}\theta)e^0,\ e^{i\phi}\sin(\tfrac{1}{2}\theta)\ ) \\ &= (\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)\end{aligned} \tag{118}$$

In the last step we identified that $\psi_1 = 0$ and $\psi_2 = \phi$. Now the surjectivity of $\pi : S^3 \longrightarrow S^2$ is easy to show. Take $(x,y,z) \in U_N \subset S^2$ then $\sigma_N((x,y,z))$ clearly maps under $\pi$ to $\pi(\sigma_N((x,y,z))) = (x,y,z)$. To cover the points in $U_S$ we use the same argument with the section $\sigma_S$. Therefore the Hopf map is surjective.