Your solutions should be neat, correct and complete. Same instructions as Mission 1 apply here.

Recommended Homework from Textbook: problems:

Chapter 13 #'s 7, 15, 17, 19, 25, 31

Recommended Homework from Recommended Textbook (Young & Freedman, 9th ed): Chapter 12 (gravitation) #'s 5, 6, 8, 9, 15, 23, 27, 32, 33, 41, 42, 50, 57, 70

Suggested Reading the following resources may be helpful:

(a.) Lectures 31, 33 as posted on the course website,

(b.) Chapter 13 of the required text.

Problem 109: (2pts) Derive Kepler's Law for a circular orbit of radius R and period T. Then, use the fact that a Martian year is 1.88 Earth years to find the mean distance to Mars. Please give your answer in astronomical units (AU).

$$\frac{mv^2}{R} = \frac{GmM}{R^2}$$

m = mass of planet
M = mass of sun
R = mean orbital radius (approximating ellipse as circle)
T = orbital period

$$v = \frac{2\pi R}{T}$$

$$v^2 = \frac{GM}{R}$$

$$\left(\frac{2\pi R}{T}\right)^2 = \frac{GM}{R} \Rightarrow \frac{R^3}{T^2} = \frac{GM}{4\pi^2} \leftarrow \text{same for all planets}$$

Comparing Earth and Mars

$$\frac{R_E^3}{T_E^2} = \frac{R_m^3}{T_m^2} \rightarrow R_m = \sqrt[3]{\frac{T_m^2}{T_E^2}}\, R_E$$

$(yr)^2$ cancel

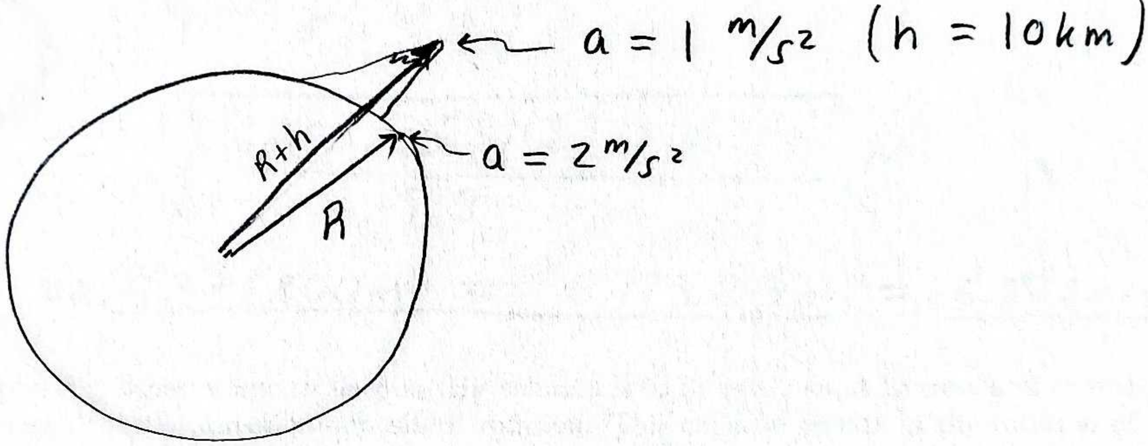
$$R_m = \sqrt[3]{(1.88)^2}\, R_E$$

$R_E = 1\,AU$

$$\boxed{R_m = 1.523\ AU}$$

Problem 110: (2pts) You measure the gravitational acceleration at a particular altitude is $2m/s^2$. After hiking $10\,km$ vertically you find the gravitational acceleration has dropped to $1\,m/s^2$. Find the mass of this mystery planet and determine your initial distance from the center of the planet.

$$a = \frac{GM}{(R+h)^2}$$ where R = radius of planet, h = altitude



$$\frac{GM}{(R+h)^2} = 1\ m/s^2 \quad \text{and} \quad \frac{GM}{R^2} = 2\ m/s^2$$

$$GM = (R+h)^2 \cdot 1\ m/s^2 = (2\ m/s^2)\,R^2$$

$$R^2 + 2Rh + h^2 = 2R^2$$

$$R^2 - 2Rh - h^2 = 0 \Rightarrow (R-h)^2 = 2h^2$$

$$R = h + h\sqrt{2} = (1+\sqrt{2})h \quad \therefore \quad \boxed{R = 24.14\ km}$$

$$M = \frac{(2\ m/s^2)(R^2)}{G} = \frac{2 \times (10^4)^2}{6.673 \times 10^{-11}} = \boxed{1.747 \times 10^{19}\ kg}$$

Problem 111: (2pts) You are given that the mass of the earth is $M_E = 5.97 \times 10^{24} kg$ and G and g take their standard values. Calculate the radius of the earth.

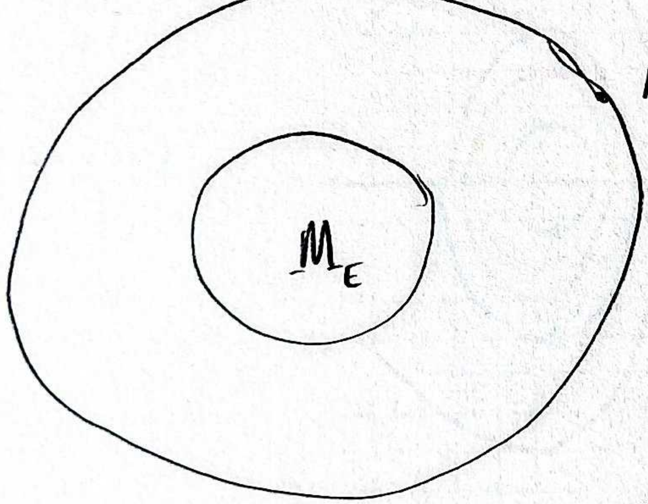
$$g = \frac{G M_E}{R_E^2} \qquad \underbrace{\left(ma = \frac{G m M_E}{R_E^2} \right)}_{\text{Newton's } 2^{nd} \text{ Law where } F_{net} = \text{gravitational force}}$$

$$R_E = \sqrt{\frac{G M_E}{g}}$$

$$= \sqrt{\frac{(6.673 \times 10^{-11})(5.97 \times 10^{24})}{9.8}} \; m$$

$$= \underline{6{,}375{,}800 m} = \underline{6375.8 \; km} = \underline{6.376 \times 10^{6} m}.$$

Problem 112: (2pts) Superman wants to sleep in. His solution is to fly into mount Everest and torque the earth in the direction opposite it rotation. This collision results in the rotation of the earth slowing to 36 hours (atomic time for you geological time silly people, I measure time on the basis of quantum mechanics not an arbitrary celestial motion). Find the new geosynchronous orbital radius.



$$\frac{G M_E m}{R^2} = \frac{m v^2}{R}$$

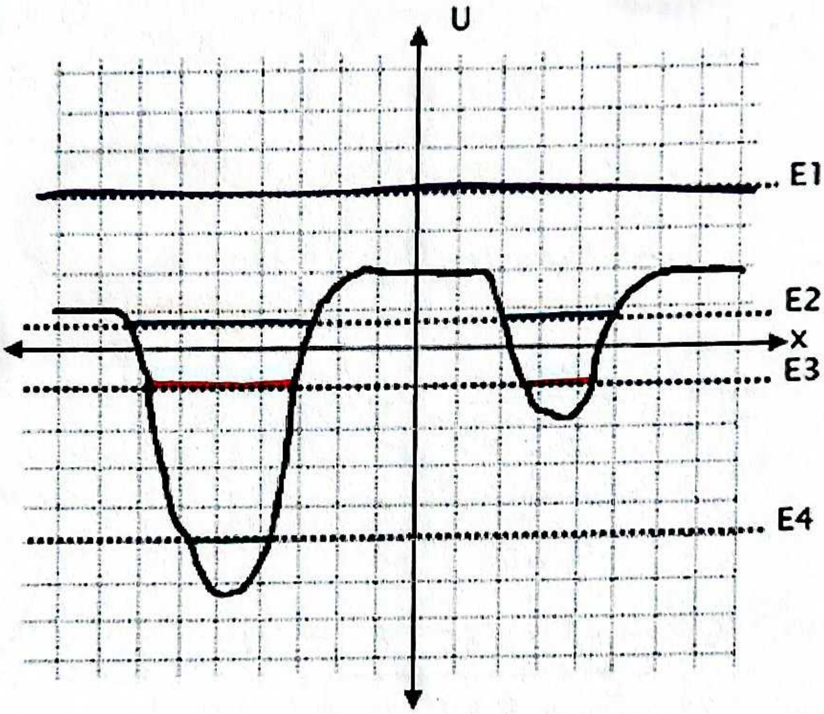
$$\frac{G M_E}{R} = v^2 = \left(\frac{2\pi R}{T}\right)^2 = \frac{4\pi^2 R^2}{T^2}$$

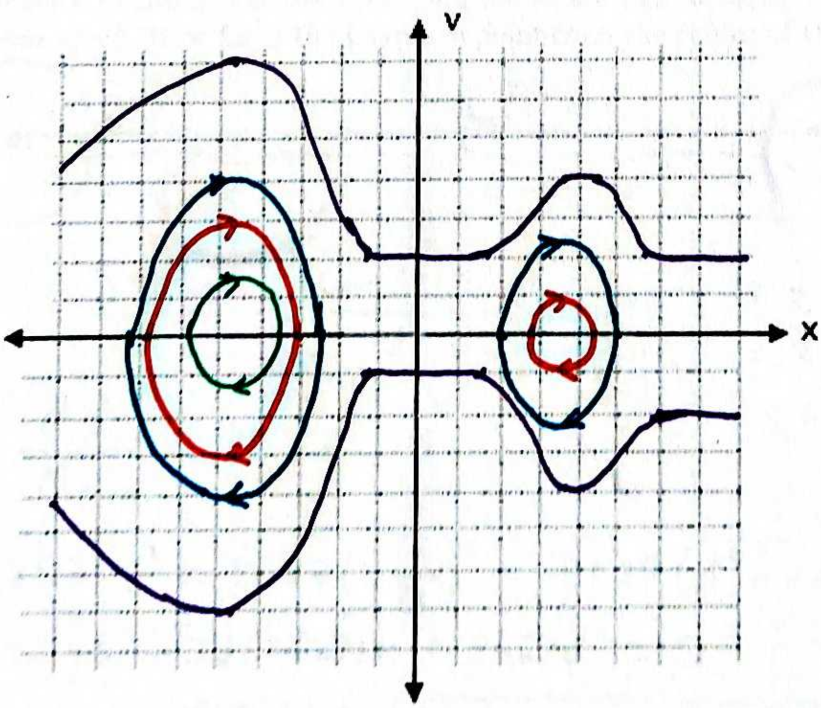
$$R^3 = \frac{G M_E T^2}{4\pi^2}$$

$$R = \sqrt[3]{\frac{(6.673 \times 10^{-11})(5.97 \times 10^{24})(1.5(86{,}400))^2}{4\pi^2}}$$

$$\boxed{R = 5.534 \times 10^{7} m \cong 55{,}341 \; km}$$

Problem 113: (2pts) Plot the motions in the xv-plane for total energy E_1, E_2, E_3, E_4 given the potential energy function plotted below. (you should plot the xv – plane beneath the given PE diagram)





(sketch... could be more careful to give consistent v for same KE values...)

Problem 114: (2pts) An ninja concentrates his considerable power and throws his opponent vertically at a speed of 10 km/s. Find the maximum height the opponent reaches. (assume these are earth-based ninjas)

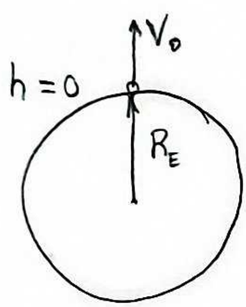
- $V_f = 0$
  $h = ?$

$$PE = \frac{-GM_E m}{R_E + h}$$

$$\left(\begin{array}{l} R_E = 6.376 \times 10^6 m \\ M_E = 5.97 \times 10^{24} kg \end{array} \right)$$

Conservation of energy,

$$\frac{-GM_E m}{R_E} + \frac{1}{2} m V_0^2 = \frac{-GM_E m}{R_E + h}$$



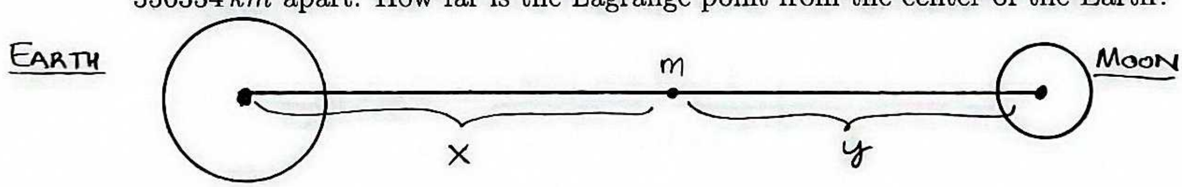
$$\frac{1}{R_E + h} = \frac{1}{R_E} - \frac{m V_0^2}{2GM_E m}$$

$$h = \frac{1}{\frac{1}{R_E} - \frac{V_0^2}{2GM_E}} - R_E$$

$$h = 3.208 \times 10^7 m - 6.376 \times 10^6 m$$

$$\boxed{h = 2.57 \times 10^7 m}$$

Problem 115: (2pts) A Lagrange point is a place where the gravitational field has a zero. Suppose for simplicity of discussion the earth and moon are placed along a line a distance of 356334 *km* apart. How far is the Lagrange point from the center of the Earth?



$$\frac{GmM_E}{x^2} = \frac{GmM_{moon}}{y^2}$$

$$x + y = 3.563 \times 10^8 m = d$$
$$M_{moon} = 7.348 \times 10^{22} kg$$
$$M_E = 5.972 \times 10^{24} kg$$

$$\frac{x^2}{y^2} = \frac{M_E}{M_{moon}} = 81.27$$

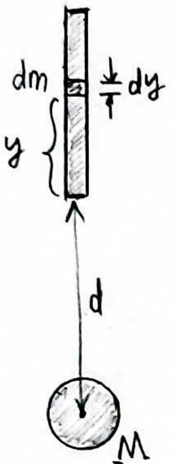
$$x^2 = 81.27y^2 = 81.27(d-x)^2 = 81.27(d^2 - 2dx + x^2)$$

$$80.27x^2 - 2d(81.27)x + 81.27d^2 = 0$$

$$x = \frac{2d(81.27) \pm \sqrt{(2d(81.27))^2 - 4(80.27)(81.27d^2)}}{2(80.27)}$$

$$\underbrace{x = 4.01 \times 10^8 m}_{\substack{x > d \\ \text{unphysical}}} \quad \text{or} \quad \boxed{x = 3.2 \times 10^8 m}$$

Problem 116: (2pts) A uniform sphere of mass M is located near a thin, uniform rod of mass m and length L. Find the force of gravity on the rod due to the sphere. (this is a calculus problem: break up the rod into infinitesimal masses, find the dF on each dm and integrate!)



$$dF = \frac{GM\,dm}{(y+d)^2} = \frac{GM\lambda\,dy}{(y+d)^2} \qquad \lambda = \frac{m}{L}$$

Here dF is the gravitational force on dm from M. To find net-force we add all the dF for $0 \le y \le L$. Let $\alpha = GM\lambda = \frac{GMm}{L}$

$$F = \int_0^L \frac{\alpha\,dy}{(y+d)^2}$$

$$= \left.\frac{-\alpha}{y+d}\right|_{y=0}^{y=L}$$

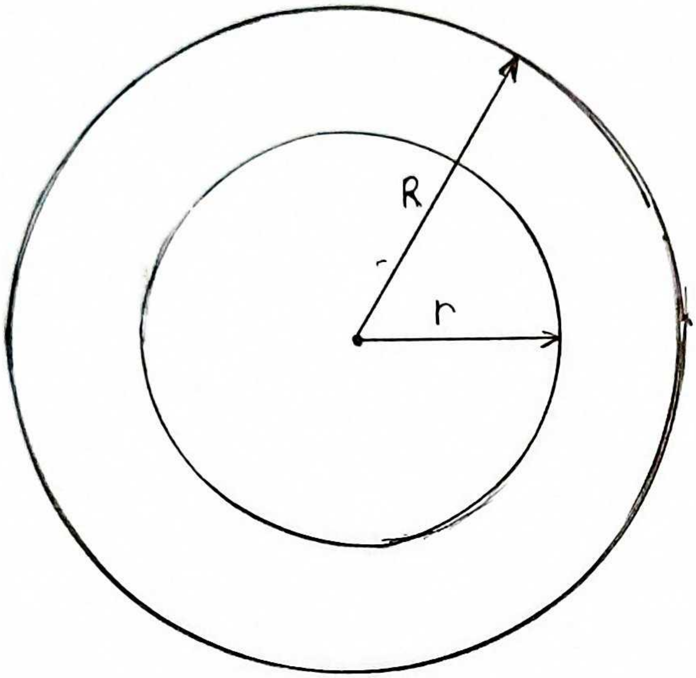
$$= \frac{-\alpha}{L+d} + \frac{\alpha}{d}$$

$$= \alpha\left(\frac{1}{d} - \frac{1}{d+L}\right)$$

$$= \frac{GMm}{L}\left(\frac{d+L-d}{d(d+L)}\right)$$

$$= \boxed{\frac{GMm}{d(d+L)}} \approx \frac{GMm}{d^2} \quad \text{for } d \gg L.$$

Problem 117: (2pts) A sphere of mass M has constant density spread over $0 \leq r \leq R$. Symmetry suggests that at a particular radius $r < R$ the gravitational acceleration is due to the mass inside the given radius. Apply this principle at arbitrary radius to find the acceleration due to gravity for the uniform sphere.



$$\underbrace{m_r = \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\right) M}$$

mass with the solid sphere of radius r

$$m_r = \frac{r^3 M}{R^3}$$

For $0 \leq r \leq R$,

$$a = \frac{G m_r}{r^2} = \frac{G}{r^2}\left(\frac{r^3 M}{R^3}\right) = \frac{GMr}{R^3}$$

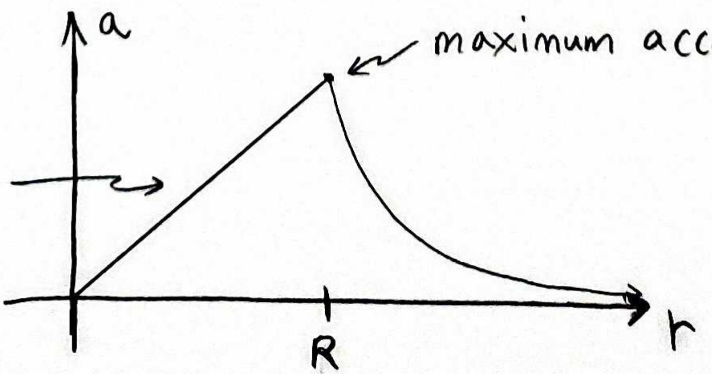
when $r \geq R$ the total mass M is inside the sphere of radius r thus

$$a = \frac{GM}{r^2}$$

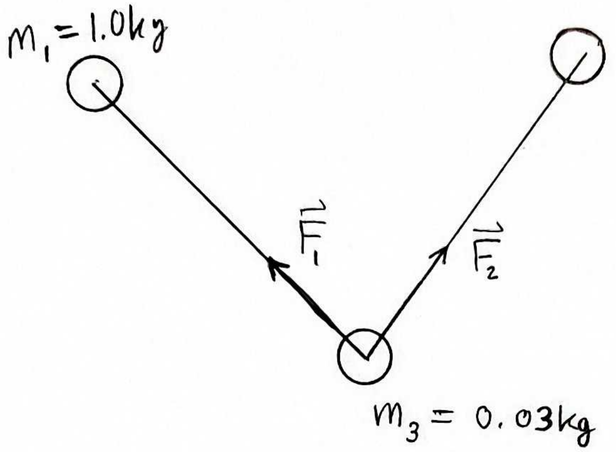
Graphically,

maximum acceleration at $r = R$.

assumes <u>uniform</u> mass density. Probably not realistic for Earth.



Problem 118: (2pts) Let masses $m_1 = 1.0\,kg$ be placed at $(1.0\,m, 0, 3.0\,m)$ and $m_2 = 2.0\,kg$ be placed at $(-1.0\,m, 2.0\,m, 0)$. Find the net gravitational force on $M = 0.030\,kg$ placed at the origin. What is the gravitational acceleration due to m_1 and m_2 at the origin?



$\vec{r}_1 = \langle 1, 0, 3 \rangle m$

$r_1 = \sqrt{10}\, m$

$\vec{r}_2 = \langle -1, 2, 0 \rangle m$

$r_2 = \sqrt{5}\, m$

$$\vec{F}_1 = \frac{Gm_1 m_3}{r_1^3}\vec{r}_1 = \frac{Gm_1 m_3}{r_1^3}\langle 1, 0, 3\rangle m$$

$$\vec{F}_2 = \frac{Gm_2 m_3}{r_2^3}\vec{r}_2 = \frac{Gm_2 m_3}{r_2^3}\langle -1, 2, 0\rangle m$$

$$m_3\vec{a} = \vec{F}_1 + \vec{F}_2$$

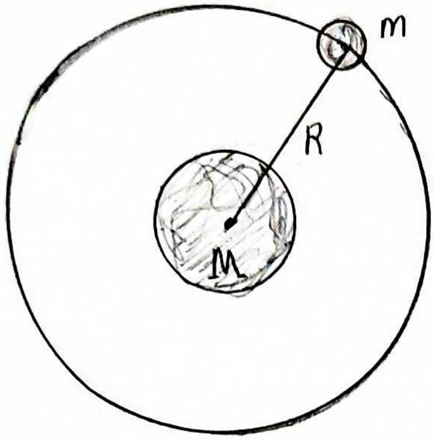
$$\vec{a} = \frac{Gm_1}{r_1^3}\langle m, 0, 3m\rangle + \frac{Gm_2}{r_2^3}\langle -1m, 2m, 0\rangle$$

$$= G\left\langle \frac{1.0kg\,m}{(\sqrt{10}\,m)^3} - \frac{(2.0kg)m}{(\sqrt{5}\,m)^3}, \frac{(2.0kg)2m}{(\sqrt{5}m)^3}, \frac{(1.0kg)(3m)}{(\sqrt{10}\,m)^3}\right\rangle$$

$$= G\left\langle -0.1473\frac{kg}{m^2}, 0.3578\frac{kg}{m^2}, 0.0949\frac{kg}{m^2}\right\rangle$$

$$= \boxed{\langle -9.829\times10^{-12}, 2.388\times10^{-11}, 6.33\times10^{-12}\rangle \frac{m}{s^2}}$$

Problem 119: (2pts) A planet has mass $M = 3.54 \times 10^{27}\ kg$. A moon orbits the planet in a circular orbit of radius $R = 2.0 \times 10^{8}\ m$. What is the period of the moon's orbit?



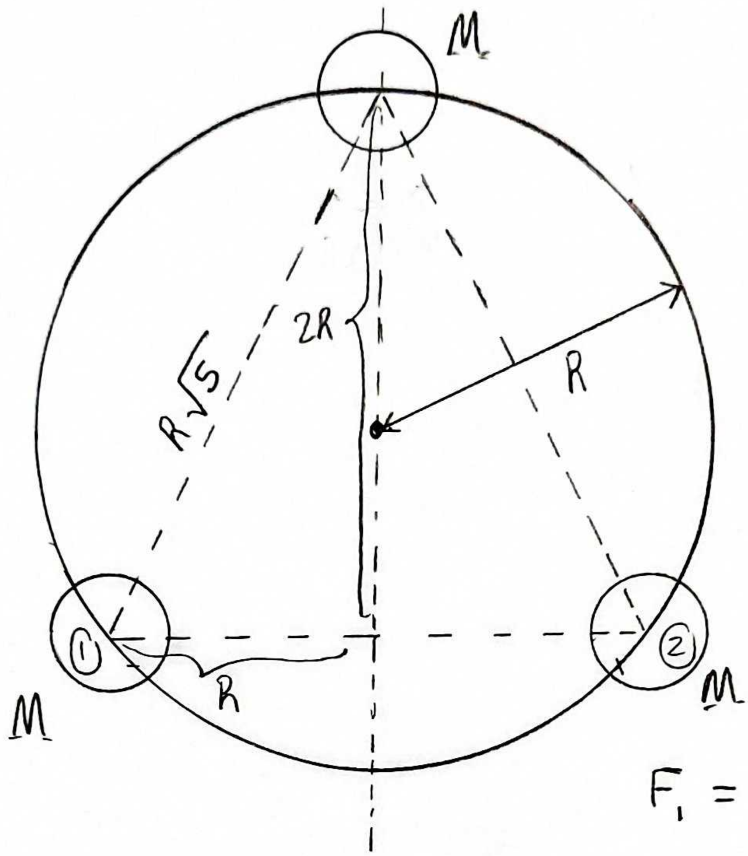
$$\frac{mv^2}{R} = \frac{GmM}{R^2} \rightarrow v^2 = \frac{GM}{R} = \left(\frac{2\pi R}{T}\right)^2$$

$$T^2 = \frac{4\pi^2 R^3}{GM}$$

$$T = \sqrt{\frac{4\pi^2 (2.0 \times 10^8 m)^3}{(6.67 \times 10^{-11} \frac{Nm^2}{kg^2})(3.54 \times 10^{27} kg)}}$$

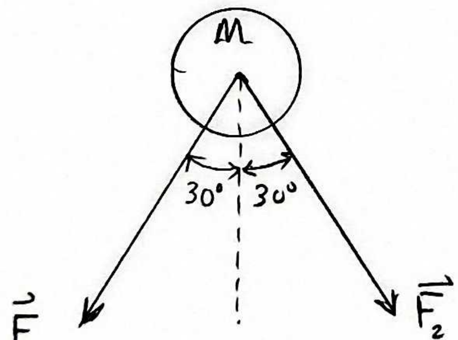
$$\boxed{T = 36{,}564.8s}$$ (aka about 10 hrs)

Problem 120: (2pts) Three planets of identical mass M orbit in a circular orbit of radius R. The planets are symmetrically placed. Find the speed of their orbit.



picture off a bit, this should be equilateral Δ

$$F_1 = \frac{GMM}{(R\sqrt{5})^2}$$



$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$ ⇐ horizonal parts cancel

$$F_{1y} = (\cos 30°)F_1 + (\cos 30°)F_2$$

$$F_{net,y} = 2\cos 30° F_1 = \sqrt{3}F_1$$

$$\frac{M.v^2}{R} = F_{net,y} = \sqrt{3}\frac{GM^2}{(R\sqrt{5})^2}$$

$$\boxed{v = \sqrt{\frac{GM\sqrt{3}}{5R}}}$$