PHYSICS 231 | MISSION 4: CONSERVATIVE FORCES & ENERGY (20+4PTS)

Your solutions should be neat, correct and complete. Same instructions as Mission 1 apply here.

Recommended Homework from Textbook (Serway):
Chapter 7 #'s 9, 11, 19, 23, 33, 38
Recommended Homework from Recommended Textbook (Young & Freedman, 9th ed):
Chapter 6 (work and kinetic energy) #'s 1, 5, 7, 9, 11, 13, 17, 21, 23, 27, 29, 33, 35, 37, 39, 47, 52, 55, 57, 59, 63, 65, 67, 71

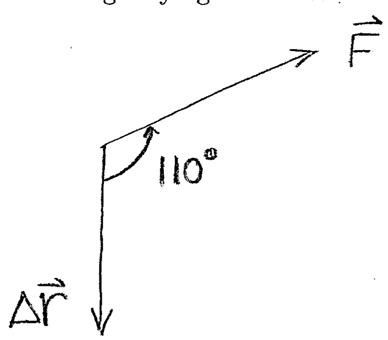
Suggested Reading the following resources may be helpful:

(a.) Lectures 15, 16, 17 and 18 as posted on the course website,

(b.) Chapter 7 of the required text.

Some people interpret this in opposite fashion, so allow ± answer for full credit here

Problem 37: (2pts) Suppose steady wind blows 20° north of east such that it places a force of 100 N on an eagle flying south. What is the work done on the eagle as it flies one mile south ?



$$W = \vec{F}\cdot\Delta\vec{r}$$
$$= (100N)(1\,\text{mile})\cos(110^\circ)$$
$$= (100N)(1\,\text{mile})\left(\frac{5280\,ft}{1\,\text{mile}}\right)\left(\frac{1\,m}{3.281\,ft}\right)\cos(110^\circ)$$
$$= \boxed{-55{,}040\,J}$$

Problem 38: (2pts) Suppose $F(x) = a - kx$ where a, k are constants. Find the potential energy function U for which $U(0) = 0$. Also, find the work done by F as a particle moves from x_1 to x_2.

$$F = a - kx = -\frac{dU}{dx}$$

$$\frac{dU}{dx} = kx - a \Rightarrow U(x) = \frac{1}{2}kx^2 - ax + C$$

But, $U(0) = 0 - 0 + C = 0 \therefore \boxed{U(x) = \frac{1}{2}kx^2 - ax}$

$$W = -\Delta PE = U(x_1) - U(x_2)$$
$$= \frac{1}{2}kx_1^2 - ax_1 - \frac{1}{2}kx_2^2 + ax_2$$
$$= \boxed{a(x_2 - x_1) + \frac{1}{2}k(x_1^2 - x_2^2)}$$

Problem 39: (3pts) Let $\vec{F} = \langle 2x, 2y - x \rangle$. Calculate the work $W = \int_C \vec{F} \cdot d\vec{r}$ done by this force field on a particle which moves along the following paths:

(a.) C is the straight line from $(1,0)$ to $(0,1)$ given by $x = 1 - t, y = t$ for $0 \leq t \leq 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(1-t, t) \cdot \frac{d}{dt}\langle 1-t, t \rangle \, dt$$

$$= \int_0^1 \langle 2(1-t), 2t - 1 + t \rangle \cdot \langle -1, 1 \rangle \, dt$$

$$= \int_0^1 (2t - 2 + 3t - 1) \, dt = \int_0^1 (5t - 3) \, dt = \frac{5}{2} - 3$$

$$= \boxed{-0.5}$$

(b.) C is the quarter-circle given by $x = \cos t, y = \sin t$ for $0 \leq t \leq \pi/2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(\cos t, \sin t) \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\pi/2} \langle 2\cos t, 2\sin t - \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\pi/2} (\cancel{-2\cos t \sin t} + \cancel{2\sin t \cos t} - \cos^2 t) \, dt$$

$$= \frac{-1}{2} \int_0^{\pi/2} (1 + \cos(2t)) \, dt = \frac{-1}{2}\left(\frac{\pi}{2}\right) = \boxed{\frac{-\pi}{4} \cong -0.7854}$$

(c.) Is $\vec{F}$ a conservative vector field on $\mathbb{R}^2$? (explain)

$\vec{F}$ is not path-independent since work done by $\vec{F}$ from $(0,1)$ to $(1,0)$ depends on path in view of (a) and (b.)

Problem 40: (1pts) For each force given below, find a potential energy function U for which $\vec{F} = -\nabla U$.

(a.) $\vec{F} = (x^2 + 1)\hat{\mathbf{x}} + \hat{\mathbf{y}} + ze^{-z^2}\hat{\mathbf{z}} = -\left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle$

$$\boxed{U(x,y,z) = \frac{-1}{3}x^3 - x - y + \frac{1}{2}e^{-z^2}}$$

(can add any constant)

(b.) $\vec{F} = \vec{F_o}$ where $\vec{F_o} = \langle a, b, c \rangle$ and a, b, c are constants.

$$\boxed{U(x,y,z) = -ax - by - cz}$$

Check it,

$$\nabla U = \langle \partial_x(-ax-by-cz), \partial_y(-ax-by-cz), \partial_z(-ax-by-cz) \rangle = \langle -a, -b, -c \rangle$$

$$\therefore \; -\nabla(-ax - by - cz) = \langle a, b, c \rangle = \vec{F_o}.$$

Problem 41: (3pts) Consider a mass $M = 20\,kg$ which moves from $(1.00\,m, -2.00\,m)$ in a straight line from to the final position $(4.00\,m, 3.00\,m)$. Find

$$\Delta\vec{r} = \big((4,3)-(1,-2)\big)\,m = \langle 3m, 5m\rangle$$

(a.) the work done by $\vec{F}_1 = \langle 10N, 0\rangle$,

$$W_1 = \vec{F}_1 \cdot \Delta\vec{r} = \langle 10N, 0\rangle \cdot \langle 3m, 5m\rangle$$
$$= 30\,Nm + 0\,Nm$$
$$= \boxed{30\,J}$$

(b.) the work done by $\vec{F}_2 = \langle 10N, 3N\rangle$,

$$W_2 = \vec{F}_2 \cdot \Delta\vec{r}$$
$$= \langle 10N, 3N\rangle \cdot \langle 3m, 5m\rangle$$
$$= (30 + 15)\,Nm = \boxed{45\,J}$$

(c.) work done by the variable force $\vec{F}_3 = (10N/m)\langle x, y\rangle$

I'll omit units for the calculation in m & s,

$$\vec{r}(t) = (1,-2) + t\big((4,3)-(1,-2)\big)$$
$$\vec{r}(t) = (1+3t, -2+5t) \quad \text{for } 0 \le t \le 1$$

Setting $x = 1+3t$ and $y = -2+5t$ parametrize the line-segment,

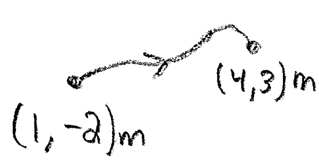
$$W = \int_0^1 10\langle 1+3t, -2+5t\rangle \cdot \langle 3, 5\rangle\,dt$$
$$= \int_0^1 10(3 + 9t - 10 + 25t)\,dt$$
$$= 10\int_0^1 (34t - 7)\,dt = 10\left(\tfrac{34}{2} - 7\right) \Rightarrow \boxed{W = 100\,J}$$

(d.) would the answers be different if the motion was not along a straight line ?

$$\text{Observe} \quad U = \left(-5\tfrac{N}{m}\right)(x^2 + y^2)$$
$$-\nabla U = 5\tfrac{N}{m}\langle 2x, 2y\rangle = 10\tfrac{N}{m}\langle x, y\rangle$$

Thus $\vec{F}$ is conservative $\Rightarrow$ path-independent.

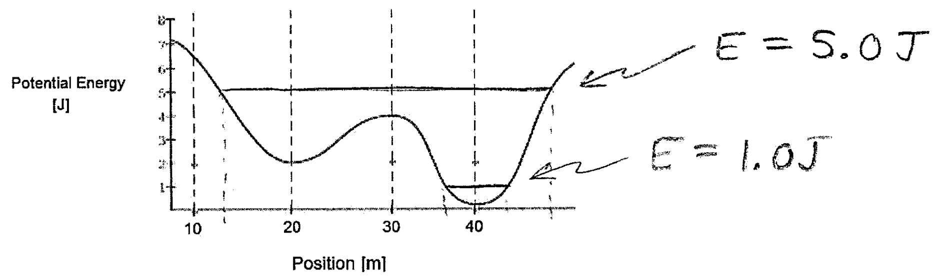
$$W_{\text{along}} = -U(4m, 3m) + U(1m, -2m)$$
$$= \left(-5\tfrac{N}{m}\right)\left[-(4m)^2 - (3m)^2 + (1m)^2 + (-2m)^2\right]$$



$$= \left(5\tfrac{N}{m}\right)\left[20\,m^2\right] = \boxed{100\,J}$$

So, no, the answers would not be different.

Problem 42: (1pts) You are given the graph of potential energy for a particle under the influence of a particular conservative force.



(a.) If the total energy of the particle of $1.0\,J$ and your initial position is $x = 40\,m$ then what is the possible range of motion (answer is approximate)

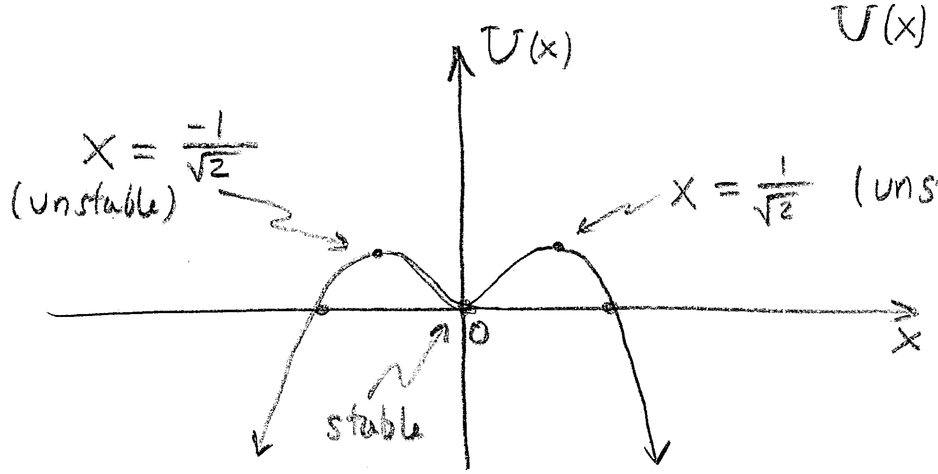
$$36m < x < 44m$$

(b.) If the total energy of the particle is $5.0\,J$ and your initial position is $x = 20\,m$ then what is the possible range of motion (answer is approximate)

$$13m < x < 50m$$

Problem 43: (2pts) Suppose $U(x) = x^2 - x^4$ is the potential energy function. Plot the energy diagram and comment on the stability of any critical points. If F is the force described by this potential energy function then explain where the force is directed right/left. Please give your answer in terms of interval notation. (for example if $2 \leq x \leq 3$ was where F points right then you would say "the force is directed to the right on $[2,3]$)

$$U(x) = x^2(1-x^2) = x^2(1+x)(1-x)$$



$$\frac{dU}{dx} = 2x - 4x^3 = 2x(1-2x^2) = 2x(1-x\sqrt{2})(1+x\sqrt{2})$$

$$\frac{dU}{dx} = 0 \Rightarrow x = 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$$

- U is increasing hence $F = -\frac{dU}{dx} < 0$ is directed leftward for the intervals $(-\infty, -1/\sqrt{2}) \cup (0, 1/\sqrt{2})$.
- Likewise F is directed rightward on $(-1/\sqrt{2}, 0) \cup (1/\sqrt{2}, \infty)$

Problem 44: (2pts) Suppose $\vec{F} = (3.2\,N)\hat{\mathbf{x}} - (6.1\,N)\hat{\mathbf{y}} + (13.1\,N)\hat{\mathbf{z}}$ acts on a mass $M = 20\,kg$ as the mass moves with constant velocity $\vec{v}(t) = \left(1.0\frac{m}{s}\right)\hat{\mathbf{x}} + \left(3.0\frac{m}{s}\right)\hat{\mathbf{y}} - \left(2.0\frac{m}{s}\right)\hat{\mathbf{z}}$. What is the power developed by the given force ? If the force is applied for the time interval $0 \leq t \leq 2.00\,s$ then what is the work done by the force on M ? What is the work done by the net-force on M ?

$$\vec{F} \cdot \vec{V} = \langle 3.2N, -6.1N, 13.1N \rangle \cdot \langle 1\tfrac{m}{s}, 3\tfrac{m}{s}, -2\tfrac{m}{s} \rangle$$
$$= \left[3.2 - (6.1)(3) - (13.1)(2)\right] N\tfrac{m}{s}$$
$$= \boxed{(-41.3)\,W} \quad \left(W = \text{Watt} = \frac{\text{Joule}}{\text{Second}} = \frac{Nm}{s} = \frac{kg\,m^2}{s^3}\right)$$

power developed by $\vec{F}$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2.00s} \left(-41.3\,\tfrac{J}{s}\right) dt$$
$$= \left(-41.3\,\tfrac{J}{s}\right)(2.00s)$$
$$= \boxed{-82.6\,J}$$

work done by $\vec{F}$ over $0 \leq t \leq 2.0s$.

$$W_{net} = \int_C \vec{F}_{net} \cdot d\vec{r} = \Delta KE$$
$$= \boxed{0}$$

work-energy Th$^{\underline{m}}$

speed constant is given.

(alternatively, $\vec{F}_{net} = m\vec{a} = \vec{0}$ so the work integral is clearly zero)

Problem 45: (2pts) We omit units here, my apologies. Consider $\vec{F}(x,y,z) = \langle 3x^2, 3y^3, -6z \rangle$.

(a.) Find the potential energy function for $\vec{F}$

(b.) Calculate the work done by $\vec{F}$ along a line-segment from $(1,2,3)$ to $(-2,0,4)$,

(c.) Calculate the work done by $\vec{F}$ along a curve which begins where it ends.

(a.) $$-\vec{F} = \nabla U = \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle$$

$$\frac{\partial U}{\partial x} = -3x^2 \longrightarrow U = -x^3 + C_1(y,z)$$

$$\frac{\partial U}{\partial y} = -3y^3 \longrightarrow U = \frac{-3}{4}y^4 + C_2(x,z)$$

$$\frac{\partial U}{\partial z} = 6z \longrightarrow U = 3z^2 + C_3(x,y)$$

We find (and I don't think you need to write all of what I wrote above)

$$\boxed{U = -x^3 - \frac{3}{4}y^4 + 3z^2}$$

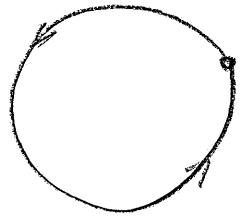
(b.) $$W = \int_{(1,2,3)}^{(-2,0,4)} \vec{F} \cdot d\vec{r} = -\Delta PE$$

$$= U(1,2,3) - U(-2,0,4)$$

$$= -1 - \frac{3}{4}(16) + 3(9) - \left(-(-2)^3 + 3(16)\right)$$

$$= \boxed{-42}$$

(c.) $$W = \oint_C \vec{F} \cdot d\vec{r} = U(P) - U(P) = \boxed{0}$$



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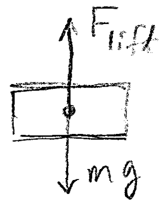
(conservative force does zero work around a loop)

Problem 46: (2pts) A 30 kg crate is lifted by a constant force at a constant velocity from the ground to a shelf 1.2 m above the ground.

(a.) What is the work done on the crate by the lifting force



$$\underbrace{ma}_{zero} = F_{lift} - mg \quad \therefore \quad F_{lift} = mg$$

$$W = F_{lift}\,\Delta y = (30kg)(9.8\tfrac{m}{s^2})(1.2m)$$
$$= \boxed{352.8\ J}$$

(b.) What is the work done on the crate by the gravitational force

$$W_{gravity} = -mg\Delta y = \boxed{-352.8\ J}$$

gravity points opposite the direction of the displacement Δy.

(c.) What is the work done on the crate by the net-force

$$W_{net} = \Delta KE = \boxed{0} \quad \text{(constant velocity)}$$

OR $$W_{net} = W_{lift} + W_{gravity} = 352.8J - 352.8J = \boxed{0}$$

(d.) What is the net change in KE for the crate.

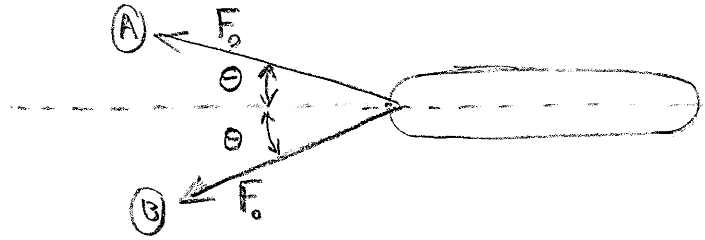
As I was just saying, $\boxed{0}$
by the fact we were
given the crate has
constant velocity.

$$KE = \frac{1}{2}mv^2$$

no change in v & m

$\Longrightarrow$ no change in KE.

Problem 47: (2pts) Two tugboats pull a disabled supertanker. Each tug exerts a constant force of $1.50 \times 10^6\ N$, one 16^o north of west and the other 16^o south of west, as the pull the tanker $0.65\ km$ toward the west. What is the total work they do on the supertanker ?



$F_0 = 1.5 \times 10^6\ N$

$\theta = 16^\circ$

$\Delta x = 0.65 km$

(taking West as positive)

$W_A = F_0 \Delta x \cos\theta$

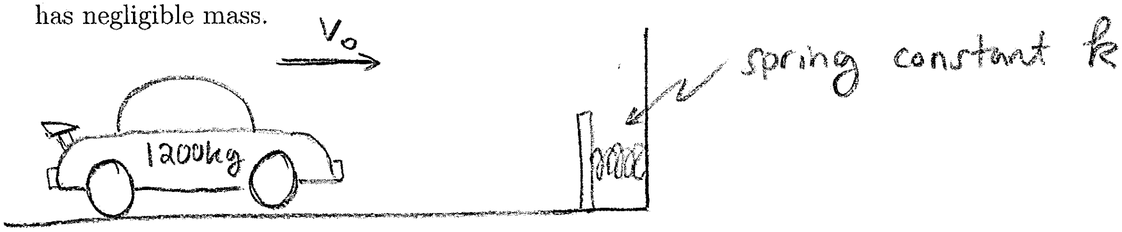
$W_B = F_0 \Delta x \cos\theta$

$$W_{TOTAL} = W_A + W_B = 2F_0 \Delta x \cos\theta$$

$$= 2(1.5\times10^6 N)(0.65\times10^3 m)\cos(16^\circ)$$

$$= \boxed{1.874 \times 10^9\ J}$$

Problem 48: (2pts) You are asked to design spring bumpers for the walls of a parking garage. A freely rolling 1200 kg car moving at 0.59 m/s is to compress the spring no more than 0.070 m before stopping. What should be the force constant of the spring ? Assume the spring has negligible mass.



conserve energy, $\frac{1}{2} m V_0^2 = \frac{1}{2} k x^2$

$$k = \frac{m V_0^2}{x^2}$$

$$= \frac{(1200 kg)(0.59\ m/s)^2}{(0.07 m)^2}$$

$$= \boxed{8.525 \times 10^4\ \frac{kg}{s^2}}$$

$$= 85284.98\ \frac{N}{m}$$